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# Riemann theta functions periodic wave solutions and rational characteristics for the nonlinear equations

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A B S T R A C T

In this paper, based on a multidimensional Riemann theta function, a lucid and straightforward generalization of the Hirota–Riemann method is presented to explicitly construct multiperiodic Riemann theta functions periodic wave solutions for nonlinear equations such as the Caudrey–Dodd–Gibbon–Sawada–Kotera equation and $(2+1)$-dimensional breaking soliton equation. Among these periodic waves, the one-periodic waves are well-known cnoidal waves, their surface pattern is one-dimensional, and often they are used as one-dimensional models of periodic waves. The two-periodic waves are a direct generalization of one-periodic waves, their surface pattern is two-dimensional so that they have two independent spatial periods in two independent horizontal directions. A limiting procedure is presented to analyze in detail, asymptotic behavior of the multiperiodic waves and the relations between the periodic wave solutions and soliton solutions are rigorously established. This generalized Hirota–Riemann method can also be demonstrated on a class variety of nonlinear difference equations such as Toeplitz lattice equation.



## 1. Introduction

Analytical studies of various linear and nonlinear, static and dynamic elasticity models, in particular, the problem of finding exact solutions, have attracted significant attention of researchers in recent years. It is always important to search for exact solutions to nonlinear differential equations, which plays an important role in the study of nonlinear physical phenomena. For example, the wave phenomena observed in fluid dynamics, plasma and elastic media are often modeled by the bell shaped such solutions and the kink shaped tanh traveling wave solutions. Different approaches, particularly in soliton theory, provide many tools for constructing explicit and exact solutions. Various kinds of exact solutions such as solitons, positons, complexitons, solitonoffs, dromions, cuspon, rational, periodic, and quasi-periodic have been presented for nonlinear integrable equations. Successful methods include the inverse scattering transform [1], Lie group [2], the Darboux transformation [3], Hirota direct method [4] and algebro-geometrical approach [5].

The algebro-geometrical approach presents quasi-periodic or algebro-geometric solutions to many soliton equations, which were originally obtained on the Korteweg–de Vries (KdV) equation based inverse spectral theory and algebro-geometric method developed by pioneers such as Novikov, Dubrovin, Mckean, Lax, Its, Matveev and co-workers [6–10] in the late 1970s. By now this theory has been extended to a large class of nonlinear integrable equations including the sine-Gordon equation, Camassa–Holm equation, Kadomtsev–Petviashvili equation, Ablowitz–Ladik lattice, and Toda lattice [5, 11–19]. All the main physical characteristics of the quasi-periodic solutions (wave numbers, phase velocities, amplitudes of the interacting modes) are defined by a compact Riemann surface. However, using the algebro-geometric theory is rather

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difficult to directly determine these characteristic parameters. In 1980s, Nakamura proposed a convenient way to construct a kind of quasi-periodic solutions of nonlinear equations in his two serial papers [20,21], where the periodic wave solutions of the KdV equation and the Boussinesq equation were obtained by means of the Hirota's bilinear method [22,23]. Recently, Hon and Fan have extended this method to investigate the discrete Toda lattice [24], $(2+1)$-dimensional Bogoyavlenskii's breaking soliton equation [25] and the asymmetrical Nizhnik–Novikov–Veselov equation [26]. Ma constructed one-periodic and two-periodic wave solutions to a class of $(2+1)$-dimensional Hirota bilinear equations [27]. Chow gave the exact periodic solutions by the Hirota bilinear method and theta functions identities for some evolution equations [28–31].

Recently, by using Darboux transformation, we obtain some periodic solutions for generalized derivative nonlinear Schrödinger equation, mKP equation with self-consistent sources and discrete soliton equations [32,33]. In this paper, we would like to give a systematic approach of generalized Hirota–Riemann method to construct multiperiodic wave solutions and rational characteristics for nonlinear equations such as the Caudrey–Dodd–Gibbon–Sawada–Kotera equation and $(2+1)$-dimensional breaking soliton equation by using multidimensional Riemann theta function. This method can also be demonstrated on a class variety of nonlinear difference equations, such as Toeplitz lattice equation, which are discussed in the conclusion.

The organization of this paper is as follows. In Section 2, we give a systematic approach of generalized Hirota–Riemann method for nonlinear equation (2.1), which briefly introduce a useful bilinear form of Eq. (2.1), the Riemann theta function and its periodicity. In Section 3, based on one-dimensional and two-dimensional Riemann theta functions, we provide two key theorems for constructing one-periodic and two-periodic wave solutions of nonlinear equation (2.1), respectively. The one-periodic wave solutions are well-known cnoidal waves and their surface pattern is one-dimensional. The two-periodic wave solutions, whose surface pattern is two-dimensional, are a direct generalization of one-periodic waves. In Sections 4 and 5, we apply this method to the Caudrey–Dodd–Gibbon–Sawada–Kotera equation and $(2+1)$-dimensional breaking soliton equation. In addition, we further analyze the features and asymptotic behavior of the one-periodic and two-periodic wave solutions in detail by making a limiting procedure, which is rigorously shown that the periodic solutions tend to the known soliton solutions under a small amplitude limit. Finally, we consider the $N$-periodic wave solutions for the nonlinear equation (2.1). Besides some other conclusions and discussions are provided.

## 2. Generalized Hirota–Riemann method for nonlinear equations

In this section, we introduce the method of Hirota bilinear operator and Riemann theta function, which plays a central role in the construction of periodic solutions of nonlinear equations.

Let us consider the most general form of nonlinear equation in $(N+1)$-dimensions

$$\mathscr{N}(u, u_t, u_{x_1}, u_{x_2}, \ldots, u_{x_N}, \ldots) = 0, \tag{2.1}$$

where $\mathscr{N}$ is a polynomial function, $t \in \mathbb{R}$ is the time variable and $x_1, x_2, \ldots, x_N$ are the space variables. In the following, we briefly introduce a useful bilinear form of Eq. (2.1) and some main points on the Riemann theta function.

### *2.1. The bilinear form of Eq. (2.1)*

The Hirota bilinear method is powerful in constructing exact solutions for a large number of nonlinear equations. Once a nonlinear equation is written in bilinear forms by a dependent variable transformation, multisoliton solutions are usually obtained. We first assume the bilinear form of the nonlinear equation (2.1) is

$$\mathscr{H}(D_{x_1}, D_{x_2}, \ldots, D_{x_N}, D_t) f(X,t) \cdot f(X,t) = 0, \quad X = (x_1, x_2, \ldots, x_N), \tag{2.2}$$

which is obtained by substituting the following dependent variable transformation into Eq. (2.1)

$$u = a\partial_\Lambda^n \ln f(X,t), \tag{2.3}$$

where $a$ is a constant, $\Lambda = x_1^{n_1} x_2^{n_2} \cdots x_N^{n_N}$ and $n = n_1 + n_2 + \cdots + n_N$. Here the bilinear operators $D_{x_1}, D_{x_2}, \ldots, D_{x_N}$ and $D_t$ are defined by

$$\begin{aligned} &D_{x_1}^m D_{x_2}^n \cdots D_{x_N}^p D_t^r f(X,t) \cdot g(X,t) \\ &\quad = (\partial_{x_1} - \partial_{x_1'})^m (\partial_{x_2} - \partial_{x_2'})^n \cdots (\partial_{x_N} - \partial_{x_N'})^p (\partial_t - \partial_{t'})^r f(X,t) \cdot g\big(X', t'\big)\big|_{X=X', t=t'}, \end{aligned} \tag{2.4}$$

with $X' = (x_1', x_2', \ldots, x_N')$. According to the identities of Hirota's bilinear operator [4,22,34,35] and Appendix C, we can obtain the following Proposition 1:

**Proposition 1.** *The Hirota bilinear operators $D_{x_1}, D_{x_2}, \dots, D_{x_N}$ and $D_t$ have properties*

$$\cosh(\delta D_t)e^{\xi_1}\cdot e^{\xi_2} = \cosh\bigl[\delta(\omega_1-\omega_2)\bigr]e^{\xi_1+\xi_2}, \tag{2.5a}$$

$$\sinh(\delta D_t)e^{\xi_1}\cdot e^{\xi_2} = \sinh\bigl[\delta(\omega_1-\omega_2)\bigr]e^{\xi_1+\xi_2}, \tag{2.5b}$$

$$e^{\delta D_t}e^{\xi_1}\cdot e^{\xi_2} = e^{\delta(\omega_1-\omega_2)}e^{\xi_1+\xi_2}, \tag{2.5c}$$

$$D_{x_1}^m D_{x_2}^n \cdots D_{x_N}^p D_t^r e^{\xi_1}\cdot e^{\xi_2} = (k_1-k_2)^m(l_1-l_2)^n\cdots(\rho_1-\rho_2)^p(\omega_1-\omega_2)^r e^{\xi_1+\xi_2}, \tag{2.5d}$$

*where $\xi_i = k_ix_1 + l_ix_2 + \cdots + \rho_ix_N + \omega_it + \varepsilon_i$, $i = 1, 2$, with $k_i, l_i, \dots, \rho_i, \omega_i$ and $\varepsilon_i$ being constants. More generally, we have*

$$\mathscr{H}(D_{x_1}, D_{x_2}, \dots, D_{x_N}, D_t)e^{\xi_1}\cdot e^{\xi_2} = \mathscr{H}(k_1-k_2, l_1-l_2, \dots, \rho_1-\rho_2, \omega_1-\omega_2)e^{\xi_1+\xi_2}, \tag{2.6}$$

*where $\mathscr{H}(D_{x_1}, D_{x_2}, \dots, D_{x_N}, D_t)$ is a polynomial about $D_{x_1}, D_{x_2}, \dots, D_{x_N}$ and $D_t$. This derivative formula will be a crucial key to our success in generating one-periodic, two-periodic and N-periodic wave solutions.*

According to the Hirota bilinear theory, Eq. (2.1) admits a one-soliton solution

$$u_1 = a\partial_\Lambda^n \ln\bigl(1+e^{\eta}\bigr), \tag{2.7}$$

where phase variable $\eta = \mu x_1 + \nu x_2 + \cdots + \Omega x_N + \gamma(\mu, \nu, \dots, \Omega)t + \delta$, $\mu, \nu, \dots, \Omega, \delta$ being constants and $\gamma(\mu, \nu, \dots, \Omega)$ is a function of $\mu, \nu, \dots, \Omega$. Similarly, the two-soliton solution takes the form

$$u_2 = a\partial_\Lambda^n \ln\bigl(1+e^{\eta_1}+e^{\eta_2}+e^{\eta_1+\eta_2+A_{12}}\bigr), \tag{2.8}$$

where phase variable $\eta_i = \mu_ix_1 + \nu_ix_2 + \cdots + \Omega_ix_N + \gamma(\mu_i, \nu_i, \dots, \Omega_i)t + \delta_i$, $i = 1, 2$, $e^{A_{12}} = \Theta(\mu_i, \nu_i, \dots, \Omega_i)$, and $\mu_i, \nu_i, \dots, \Omega_i, \delta_i$, $i = 1, 2$, are free constants, $\gamma(\mu_i, \nu_i, \dots, \Omega_i)$ is a function of $\mu_i, \nu_i, \dots, \Omega_i$.

Using the Hirota bilinear method to construct multiperiodic wave solutions of Eq. (2.1), we consider a slightly generalized form of the bilinear equation (2.2). Assuming Eq. (2.1) has the nonzero asymptotic condition $u \to u_0$ as $|\xi| \to 0$, we look for its solutions in the form of

$$u = u_0 + a\partial_\Lambda^n \ln\vartheta(\xi), \tag{2.9}$$

where $u_0$ is a constant solution of Eq. (2.1) and phase variable $\xi$ is taken as the form $\xi = (\xi_1, \dots, \xi_N)^T$, $\xi_i = k_ix_1 + l_ix_2 + \cdots + \rho_ix_N + \omega_it + \varepsilon_i$, $i = 1, 2, \dots, N$.

Substituting (2.9) into (2.1) and integrating with respect to $x_1$, we can obtain the following bilinear form

$$\mathscr{L}(D_{x_1}, D_{x_2}, \dots, D_{x_N}, D_t)\vartheta(\xi)\cdot\vartheta(\xi) = \mathscr{H}(D_{x_1}, D_{x_2}, \dots, D_{x_N}, D_t, c)\vartheta(\xi)\cdot\vartheta(\xi) = 0, \tag{2.10}$$

where $c = c(x_2, \dots, t)$ is an integration constant. For the bilinear equation (2.10), we are interested in its multiperiodic solutions in terms of the Riemann theta function $\vartheta(\xi)$.

**Remark 1.** The constant $c = c(x_2, \dots, t)$ may be taken to be zero in the construction of soliton solutions. But in our present periodic case, the nonzero constant $c$ plays an important role and must not be dropped. Because elliptic functions generally do not satisfy equations with zero integration constants such as Eq. (2.2).

### *2.2. The theta function*

In the following, we introduce the periodicity of the theta function $\vartheta(\xi, \tau)$. Based on Ref. [36], we can obtain Proposition 2 as follows:

**Proposition 2.** *The theta function $\vartheta(\xi, \tau)$ has the periodic properties*

$$\vartheta(\xi + 1 + \tau) = e^{-\pi i\tau - 2\pi i\xi}\vartheta(\xi, \tau). \tag{2.11}$$

*We regard the vectors 1 and $\tau$ as periods of the theta function $\vartheta(\xi, \tau)$ with multipliers 1 and $e^{-\pi i\tau-2\pi i\xi}$, respectively. Here, $\tau$ is not a period of theta function $\vartheta(\xi, \tau)$, but it is the period of the functions $\partial_\xi^2 \ln\vartheta(\xi, \tau)$, $\partial_\xi \ln[\vartheta(\xi+e, \tau)/\vartheta(\xi+h, \tau)]$ and $\vartheta(\xi+e, \tau)\vartheta(\xi-e, \tau)/\vartheta^2(\xi+h, \tau)$.*

To construct multiperiodic wave solutions of Eq. (2.1), we consider the following multidimensional Riemann theta function of genus $N$

$$\vartheta(\xi) = \vartheta(\xi, \tau) = \sum_{n\in\mathbb{Z}^N} e^{\pi i\langle n\tau, n\rangle + 2\pi i\langle \xi, n\rangle}, \tag{2.12}$$

where the integer value vector $n = (n_1, \dots, n_N)^T \in \mathbb{Z}^N$ and complex phase variables $\xi = (\xi_1, \dots, \xi_N)^T \in \mathbb{C}^N$. Moreover, for two vectors $f = (f_1, \dots, f_N)^T$ and $g = (g_1, \dots, g_N)^T$, their inner product is defined by

$$\langle f, g \rangle = f_1 g_1 + f_2 g_2 + \cdots + f_N g_N. \tag{2.13}$$

The equality of $-i\tau = -i(\tau_{ij})$ is a positive definite and real-valued symmetric $N \times N$ matrix, which we call the period matrix of the theta function. The entries $\tau_{ij}$ of the period matrix can be considered as free parameters of the theta function (2.12). Under these conditions, the Fourier series (2.12) converges to a real-valued function for an arbitrary vector $\xi \in \mathbb{C}^N$.

**Remark 2.** Construct periodic wave solution can use an algebro-geometric method [5–19] and other methods [50–58] and so on. The matrix $\tau$ is usually constructed via a compact Riemann surface $\mathscr{J}$ of genus $N \in \mathbb{N}$. In this paper, we take the matrix $\tau$ to be pure imaginary matrix to make the theta function (2.12) real valued.

## 3. Periodic wave solutions for nonlinear equations

In this section, we mainly consider how to construct the one-periodic and two-periodic wave solutions of Eq. (2.1) by using the Hirota–Riemann method.

### 3.1. Construction of one-periodic waves

In the following, we consider one-periodic wave solutions of Eq. (2.1), i.e. we first consider the simple case when $N = 1$, then Riemann theta function (2.12) reduces the following Fourier series in $n$

$$\vartheta(\xi, \tau) = \sum_{n=-\infty}^{+\infty} e^{\pi i n^2 \tau + 2\pi i n \xi}, \tag{3.1}$$

where the phase variable $\xi = kx_1 + lx_2 + \cdots + \rho x_N + \omega t + \varepsilon$ and the parameter $\mathrm{Im}(\tau) > 0$.

**Theorem 1.** *Assuming that $\vartheta(\xi, \tau)$ is one Riemann theta function as $N = 1$ with $\xi = kx_1 + lx_2 + \cdots + \rho x_N + \omega t + \varepsilon$ and $k, l, \dots, \rho, \omega, \varepsilon$ satisfy the following system*

$$\sum_{n=-\infty}^{\infty} \mathscr{L}(4n\pi ik, 4n\pi il, \dots, 4n\pi i\rho, 4n\pi i\omega) e^{2n^2\pi i\tau} = 0, \tag{3.2a}$$

$$\sum_{n=-\infty}^{\infty} \mathscr{L}\big(2\pi i(2n-1)k, 2\pi i(2n-1)l, \dots, 2\pi i(2n-1)\rho, 2\pi i(2n-1)\omega\big) e^{(2n^2-2n+1)\pi i\tau} = 0, \tag{3.2b}$$

*the following expression*

$$u = u_0 + a\partial_\Lambda^n \ln \vartheta(\xi), \tag{3.3}$$

*is the one-periodic wave solution of Eq.* (2.1)*.*

**Proof.** To make the theta function (3.1) satisfies the bilinear equation (2.10), we substitute function (3.1) into the left of Eq. (2.10) and by using the property Proposition 1 (2.5) obtain that

$$\begin{aligned}
&\mathscr{L}(D_{x_1}, D_{x_2}, \dots, D_{x_N}, D_t)\vartheta(\xi) \cdot \vartheta(\xi) \\
&\quad = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathscr{L}(D_{x_1}, D_{x_2}, \dots, D_{x_N}, D_t) e^{\pi i m^2 \tau + 2\pi i m \xi} \cdot e^{\pi i n^2 \tau + 2\pi i n \xi} \\
&\quad = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathscr{L}\big(2\pi i(n-m)k, 2\pi i(n-m)l, \dots, 2\pi i(n-m)\rho, 2\pi i(n-m)\omega\big) e^{\pi i(m^2+n^2)\tau + 2\pi i(m+n)\xi} \\
&\quad = \sum_{m'=-\infty}^{\infty} \Bigg\{ \sum_{n=-\infty}^{\infty} \mathscr{L}\big(2\pi i\big(2n-m'\big)k, 2\pi i\big(2n-m'\big)l, \dots, 2\pi i\big(2n-m'\big)\rho, 2\pi i\big(2n-m'\big)\omega\big) \\
&\qquad \times e^{\pi i[n^2+(n-m')^2]\tau} \Bigg\} e^{2\pi i m' \xi} \\
&\quad \triangleq \sum_{m'=-\infty}^{\infty} \tilde{\mathscr{L}}\big(m'\big) e^{2\pi i m' \xi}, \quad m' = m + n.
\end{aligned} \tag{3.4}$$

In the following, we compute each series $\tilde{\mathscr{L}}(m')$ for $m' \in \mathbb{Z}$. By shifting summation index by $n' = n - 1$, we have the following fact

$$
\begin{aligned}
\tilde{\mathscr{L}}(m') &= \sum_{n=-\infty}^{\infty} \mathscr{L}\big(2\pi i(2n-m')k, 2\pi i(2n-m')l, \ldots, 2\pi i(2n-m')\rho, 2\pi i(2n-m')\omega\big) e^{\pi i[n^2+(n-m')^2]\tau} \\
&= \sum_{n=-\infty}^{\infty} \mathscr{L}\big(2\pi i\big[2n' - \big(m'-2\big)\big]k, 2\pi i\big[2n' - \big(m'-2\big)\big]l, \ldots, 2\pi i\big(2n' - \big(m'-2\big)\big)\rho, 2\pi i\big(2n' - \big(m'-2\big)\big)\omega\big) \\
&\quad \times e^{\pi i\{n'^2+[n'-(m'-2)]^2\}\tau} \cdot e^{2\pi i(m'-1)\tau} \\
&= \tilde{\mathscr{L}}\big(m'-2\big) e^{2\pi i(m'-1)\tau} = \cdots = \begin{cases} \tilde{\mathscr{L}}(0) e^{\pi i m'\tau}, & m' \text{ is even}, \\ \tilde{\mathscr{L}}(1) e^{\pi i(m'+1)\tau}, & m' \text{ is odd}, \end{cases} \quad m', n' \in \mathbb{Z},
\end{aligned} \tag{3.5}
$$

which implies that $\tilde{\mathscr{L}}(m'), m \in \mathbb{Z}$ are completely dominated by two functions $\tilde{\mathscr{L}}(0)$ and $\tilde{\mathscr{L}}(1)$. If $\tilde{\mathscr{L}}(0) = \tilde{\mathscr{L}}(1) = 0$, then it follows that $\tilde{\mathscr{L}}(m') = 0$, $m' \in \mathbb{Z}$ and thus the theta function (3.1) is an exact solution to Eq. (2.10), namely, $\mathscr{L}(D_{x_1}, D_{x_2}, \ldots, D_{x_N}, D_t)\vartheta(\xi) \cdot \vartheta(\xi) = 0$. Noticing the specific form of Eq. (2.10), one-periodic wave solutions can be obtained, if we require

$$
\tilde{\mathscr{L}}(0) = \sum_{n=-\infty}^{\infty} \mathscr{L}(4n\pi ik, 4n\pi il, \ldots, 4n\pi i\rho, 4n\pi i\omega) e^{2n^2\pi i\tau} = 0, \tag{3.6a}
$$

$$
\tilde{\mathscr{L}}(1) = \sum_{n=-\infty}^{\infty} \mathscr{L}\big(2\pi i(2n-1)k, 2\pi i(2n-1)l, \ldots, 2\pi i(2n-1)\rho, 2\pi i(2n-1)\omega\big) e^{(2n^2-2n+1)\pi i\tau} = 0. \tag{3.6b}
$$

Now solving this system, we get a one-periodic wave solution of Eq. (2.1)

$$
u = u_0 + a\partial_\Lambda^n \ln \vartheta(\xi),
$$

which provided the vector $(\omega, c)^T$ solves Eq. (3.6) with the theta function $\vartheta(\xi)$ given by Eq. (3.1) and parameters $\omega$, $c$ by Eq. (3.6). The other parameters $k, l, \ldots, \rho, \tau, \varepsilon$ and $u_0$ are free. The $N+1$ parameters $k, l, \ldots, \rho$ and $\tau$ completely dominate a one-periodic wave. □

**Remark 3.** Theorem 1 actually provides us a unified approach to construct one-periodic wave solutions for nonlinear equations. Once an equation is written bilinear forms, its one-periodic wave solutions can be directly obtained by solving system (3.2).

### *3.2. Construction of two-periodic waves*

In this section, we consider two-periodic wave solutions of Eq. (2.1), which are two-dimensional generalization of one-periodic wave solutions. The two-periodic waves of interest here have three-dimensional velocity fields and two-dimensional surface patterns.

Let us now consider the case of $N = 2$ and the Riemann theta function (2.12) takes the form

$$
\vartheta(\xi, \tau) = \vartheta(\xi_1, \xi_2, \tau) = \sum_{n\in\mathbb{Z}^2} e^{\pi i\langle \tau n, n\rangle + 2\pi i\langle \xi, n\rangle}, \tag{3.7}
$$

where $n = (n_1, n_2)^T \in \mathbb{Z}^2$, $\xi = (\xi_1, \xi_2) \in \mathbb{C}^2$, $\xi_i = k_i x_1 + l_i x_2 + \cdots + \rho_i x_N + \omega_i t + \varepsilon_i$, $i = 1, 2$, and $-i\tau$ is a positive definite and real-valued symmetric $2 \times 2$ matrix which can take the form of

$$
\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix}, \quad \operatorname{Im}(\tau_{11}) > 0,\ \operatorname{Im}(\tau_{22}) > 0,\ \tau_{11}\tau_{22} - \tau_{12}^2 < 0. \tag{3.8}
$$

**Theorem 2.** *Assuming that $\vartheta(\xi_1, \xi_2, \tau)$ is one Riemann theta function as $N = 2$ with $\xi_i = k_i x_1 + l_i x_2 + \cdots + \rho_i x_N + \omega_i t + \varepsilon_i$, $i = 1, 2$, and $k_i, l_i, \ldots, \rho_i, \omega_i, \varepsilon_i$ $(i = 1, 2)$ satisfy the following system*

$$
\begin{aligned}
&\sum_{n\in\mathbb{Z}^2} \mathscr{L}\big(2\pi i\langle 2n-\theta_1, k\rangle, 2\pi i\langle 2n-\theta_1, l\rangle, \ldots, 2\pi i\langle 2n-\theta_1, \rho\rangle, 2\pi i\langle 2n-\theta_1, \omega\rangle\big) \\
&\quad \times e^{\pi i[\langle \tau(n-\theta_1), n-\theta_1\rangle + \langle \tau n, n\rangle]} = 0,
\end{aligned} \tag{3.9a}
$$

$$
\begin{aligned}
&\sum_{n\in\mathbb{Z}^2} \mathscr{L}\big(2\pi i\langle 2n-\theta_2, k\rangle, 2\pi i\langle 2n-\theta_2, l\rangle, \ldots, 2\pi i\langle 2n-\theta_2, \rho\rangle, 2\pi i\langle 2n-\theta_2, \omega\rangle\big) \\
&\quad \times e^{\pi i[\langle \tau(n-\theta_2), n-\theta_2\rangle + \langle \tau n, n\rangle]} = 0,
\end{aligned} \tag{3.9b}
$$

$$\sum_{n\in\mathbb{Z}^2}\mathscr{L}\big(2\pi i\langle 2n-\theta_3,k\rangle,2\pi i\langle 2n-\theta_3,l\rangle,\ldots,2\pi i\langle 2n-\theta_3,\rho\rangle,2\pi i\langle 2n-\theta_3,\omega\rangle\big)$$
$$\times e^{\pi i[\langle\tau(n-\theta_3),n-\theta_3\rangle+\langle\tau n,n\rangle]}=0, \tag{3.9c}$$
$$\sum_{n\in\mathbb{Z}^2}\mathscr{L}\big(2\pi i\langle 2n-\theta_4,k\rangle,2\pi i\langle 2n-\theta_4,l\rangle,\ldots,2\pi i\langle 2n-\theta_4,\rho\rangle,2\pi i\langle 2n-\theta_4,\omega\rangle\big)$$
$$\times e^{\pi i[\langle\tau(n-\theta_4),n-\theta_4\rangle+\langle\tau n,n\rangle]}=0, \tag{3.9d}$$

*where* $\theta_i=(\theta_i^1,\theta_i^2)^T$, $\theta_1=(0,0)^T$, $\theta_2=(1,0)^T$, $\theta_3=(0,1)^T$, $\theta_4=(1,1)^T$, $i=1,2,3,4$, *the following expression*

$$u=u_0+a\partial_\Lambda^n\ln\vartheta(\xi_1,\xi_2), \tag{3.10}$$

*is the two-periodic wave solution of Eq.* (2.1)*.*

**Proof.** In order to get some sufficient conditions, such that the theta function (3.7) satisfies the bilinear equation (2.10), we substitute the function (3.7) into the left of Eq. (2.10) and obtain that

$$\begin{aligned}
&\mathscr{L}(D_{x_1},D_{x_2},\ldots,D_{x_N},D_t)\vartheta(\xi_1,\xi_2,\tau)\cdot\vartheta(\xi_1,\xi_2,\tau)\\
&=\sum_{m,n\in\mathbb{Z}^2}\mathscr{L}\big(2\pi i\langle n-m,k\rangle,2\pi i\langle n-m,l\rangle,\ldots,2\pi i\langle n-m,\rho\rangle,2\pi i\langle n-m,\omega\rangle\big)e^{2\pi i\langle\xi,m+n\rangle+\pi i(\langle\tau m,m\rangle+\langle\tau n,n\rangle)}\\
&=\sum_{m'\in\mathbb{Z}^2}\Big\{\sum_{n\in\mathbb{Z}^2}\mathscr{L}\big(2\pi i\langle 2n-m',k\rangle,2\pi i\langle 2n-m',l\rangle,\ldots,2\pi i\langle 2n-m',\rho\rangle,2\pi i\langle 2n-m',\omega\rangle\big)\\
&\quad\times e^{\pi i[\langle\tau(n-m'),n-m'\rangle+\langle\tau n,n\rangle]}\Big\}e^{2\pi i\langle\xi,m'\rangle}\\
&\triangleq\sum_{m'\in\mathbb{Z}^2}\tilde{\mathscr{L}}(m_1',m_2')e^{2\pi i\langle\xi,m'\rangle}=\sum_{m'\in\mathbb{Z}^2}\tilde{\mathscr{L}}(m')e^{2\pi i\langle\xi,m'\rangle},\quad m'=m+n.
\end{aligned}\tag{3.11}$$

Shifting index $n$ as $n'=n-\delta_{ij}$, $j=1,2$, we can compute that

$$\begin{aligned}
\tilde{\mathscr{L}}(m')&=\tilde{\mathscr{L}}(m_1',m_2')=\sum_{n\in\mathbb{Z}^2}\mathscr{L}\big(2\pi i\langle 2n-m',k\rangle,2\pi i\langle 2n-m',l\rangle,\ldots,2\pi i\langle 2n-m',\rho\rangle,2\pi i\langle 2n-m',\omega\rangle\big)\\
&\quad\times e^{\pi i[\langle\tau(n-m'),n-m'\rangle+\langle\tau n,n\rangle]}\\
&=\sum_{n\in\mathbb{Z}^2}\mathscr{L}\Bigg(2\pi i\sum_{i=1}^{2}\big[2n_i'-\big(m_i'-2\delta_{ij}\big)\big]k_i,2\pi i\sum_{i=1}^{2}\big[2n_i'-\big(m_i'-2\delta_{ij}\big)\big]l_i,\ldots,2\pi i\sum_{i=1}^{2}\big[2n_i'-\big(m_i'-2\delta_{ij}\big)\big]\rho_i,\\
&\quad 2\pi i\sum_{i=1}^{2}\big[2n_i'-\big(m_i'-2\delta_{ij}\big)\big]\omega_i\Bigg)e^{\pi i\sum_{i,k=1}^{2}[(n_i'+\delta_{ij})(n_k'+\delta_{kj})+(m_i'-n_i'-\delta_{ij})(m_k'-n_k'-\delta_{kj})]\tau_{ik}}\\
&=\begin{cases}\tilde{\mathscr{L}}(m_1'-2,m_2')e^{2\pi i(m_1'-1)\tau_{11}+2\pi i m_2'\tau_{12}}, & j=1,\\ \tilde{\mathscr{L}}(m_1',m_2'-2)e^{2\pi i(m_2'-1)\tau_{22}+2\pi i m_1'\tau_{12}}, & j=2,\end{cases}\quad m',n'\in\mathbb{Z}^2,
\end{aligned}\tag{3.12}$$

where $\delta_{ij}$ representing Kronecker's delta. It implies that $\tilde{\mathscr{L}}(m')$, $m\in\mathbb{Z}^2$ are completely dominated by four functions $\tilde{\mathscr{L}}(0,0)$, $\tilde{\mathscr{L}}(1,0)$, $\tilde{\mathscr{L}}(0,1)$ and $\tilde{\mathscr{L}}(1,1)$. If $\tilde{\mathscr{L}}(0,0)=\tilde{\mathscr{L}}(1,0)=\tilde{\mathscr{L}}(0,1)=\tilde{\mathscr{L}}(1,1)=0$, then it follows that $\tilde{\mathscr{L}}(m')=0$, $m'\in\mathbb{Z}$ and thus the theta function (3.7) is an exact solution to Eq. (2.10), namely, $\mathscr{L}(D_{x_1},D_{x_2},\ldots,D_{x_N},D_t)\vartheta(\xi)\cdot\vartheta(\xi)=0$. Noticing the specific form of Eq. (2.10), two-periodic wave solutions can be obtained, if we require

$$\tilde{\mathscr{L}}(0,0)=\sum_{n\in\mathbb{Z}^2}\mathscr{L}\big(2\pi i\langle 2n-\theta_1,k\rangle,2\pi i\langle 2n-\theta_1,l\rangle,\ldots,2\pi i\langle 2n-\theta_1,\rho\rangle,2\pi i\langle 2n-\theta_1,\omega\rangle\big)$$
$$\times e^{\pi i[\langle\tau(n-\theta_1),n-\theta_1\rangle+\langle\tau n,n\rangle]}=0, \tag{3.13a}$$
$$\tilde{\mathscr{L}}(1,0)=\sum_{n\in\mathbb{Z}^2}\mathscr{L}\big(2\pi i\langle 2n-\theta_2,k\rangle,2\pi i\langle 2n-\theta_2,l\rangle,\ldots,2\pi i\langle 2n-\theta_2,\rho\rangle,2\pi i\langle 2n-\theta_2,\omega\rangle\big)$$
$$\times e^{\pi i[\langle\tau(n-\theta_2),n-\theta_2\rangle+\langle\tau n,n\rangle]}=0, \tag{3.13b}$$
$$\tilde{\mathscr{L}}(0,1)=\sum_{n\in\mathbb{Z}^2}\mathscr{L}\big(2\pi i\langle 2n-\theta_3,k\rangle,2\pi i\langle 2n-\theta_3,l\rangle,\ldots,2\pi i\langle 2n-\theta_3,\rho\rangle,2\pi i\langle 2n-\theta_3,\omega\rangle\big)$$
$$\times e^{\pi i[\langle\tau(n-\theta_3),n-\theta_3\rangle+\langle\tau n,n\rangle]}=0, \tag{3.13c}$$

$$\tilde{\mathscr{L}}(1,1)=\sum_{n\in\mathbb{Z}^2}\mathscr{L}\big(2\pi i\langle 2n-\theta_4,k\rangle,2\pi i\langle 2n-\theta_4,l\rangle,\dots,2\pi i\langle 2n-\theta_4,\rho\rangle,2\pi i\langle 2n-\theta_4,\omega\rangle\big)$$
$$\times e^{\pi i[\langle\tau(n-\theta_4),n-\theta_4\rangle+\langle\tau n,n\rangle]}=0, \tag{3.13d}$$

where $\theta_i=(\theta_i^1,\theta_i^2)^T$, $\theta_1=(0,0)^T$, $\theta_2=(1,0)^T$, $\theta_3=(0,1)^T$, $\theta_4=(1,1)^T$, $i=1,2,3,4$.

Now solving this system, we get a two-periodic wave solution of Eq. (2.1)

$$u=u_0+a\partial_\Lambda^n\ln\vartheta(\xi_1,\xi_2,\tau), \tag{3.14}$$

where $\vartheta(\xi_1,\xi_2,\tau)$ and parameters $\omega_1$, $\omega_2$, $u_0$, $c$ are given by Eqs. (3.7) and (3.13), respectively. The other parameters $k_1,k_2,l_1,l_2,\dots,\rho_1,\rho_2,\varepsilon_1,\varepsilon_2,\tau_{11},\tau_{12}$ and $\tau_{22}$ are free. The two-periodic wave is specified by $2N+3$ of the parameters $k_1,k_2,l_1,l_2,\dots,\rho_1,\rho_2,\tau_{11},\tau_{12}$ and $\tau_{22}$. □

**Remark 4.** Theorem 2 actually provides us a unified approach to construct two-periodic wave solutions for nonlinear equations. Once an equation is written bilinear forms, its two-periodic wave solutions can be directly obtained by solving system (3.9).

## 4. The Caudrey–Dodd–Gibbon–Sawada–Kotera equation

The Caudrey–Dodd–Gibbon–Sawada–Kotera equation (CDGSK) reads

$$u_t+u_{xxxxx}+30uu_{xxx}+30u_xu_{xx}+180u_{xx}^2u_x=0, \tag{4.1}$$

which is given by Sawada and Kotera [37], Dodd and Gibbon [38]. Its physical understanding was illustrated in [39]. Lou obtain twelve sets of symmetries of the CDGSK equation [40]. Wazwaz obtained soliton solutions of the equation by means of bilinear method [41]. In this section we construct its periodic wave solutions and show that the soliton solutions can be obtained as limiting case of the periodic solutions.

Applying the Hirota–Riemann method to construct periodic wave solutions of Eq. (4.1), we consider a variable transformation

$$u=u_0+2\partial_x^2\ln\vartheta(\xi), \tag{4.2}$$

where $\xi=kx+\omega t+\varepsilon$. Integrating with respect to $x$, Eq. (2.10) becomes the following bilinear form

$$\mathscr{L}(D_x,D_t)\vartheta(\xi)\cdot\vartheta(\xi)=\big(D_tD_x+30u_0D_x^4+D_x^6+c\big)\vartheta(\xi)\cdot\vartheta(\xi)=0. \tag{4.3}$$

According to Eqs. (2.7)–(2.8), Eq. (4.1) admits a one-soliton solution and two-soliton solution

$$u_1=2\partial_x^2\ln\big(1+e^{\eta}\big), \tag{4.4a}$$
$$u_2=a\partial_x^2\ln\big(1+e^{\eta_1}+e^{\eta_2}+e^{\eta_1+\eta_2+A_{12}}\big), \tag{4.4b}$$

where phase variable $\eta=\mu x-\mu^5t+\delta$, $\eta_i=\mu_ix-\mu_i^5t+\delta_i$, $i=1,2$, $e^{A_{12}}=\frac{(\mu_1+\mu_2)(\mu_1^3+2\mu_1^2\mu_2+2\mu_1\mu_1^2+\mu_2^3)}{(\mu_1-\mu_2)(\mu_1^3-2\mu_1^2\mu_2+2\mu_1\mu_1^2-\mu_2^3)}$, $\mu$, $\delta$, $\mu_i$, $\delta_i$, $i=1,2$, are constants.

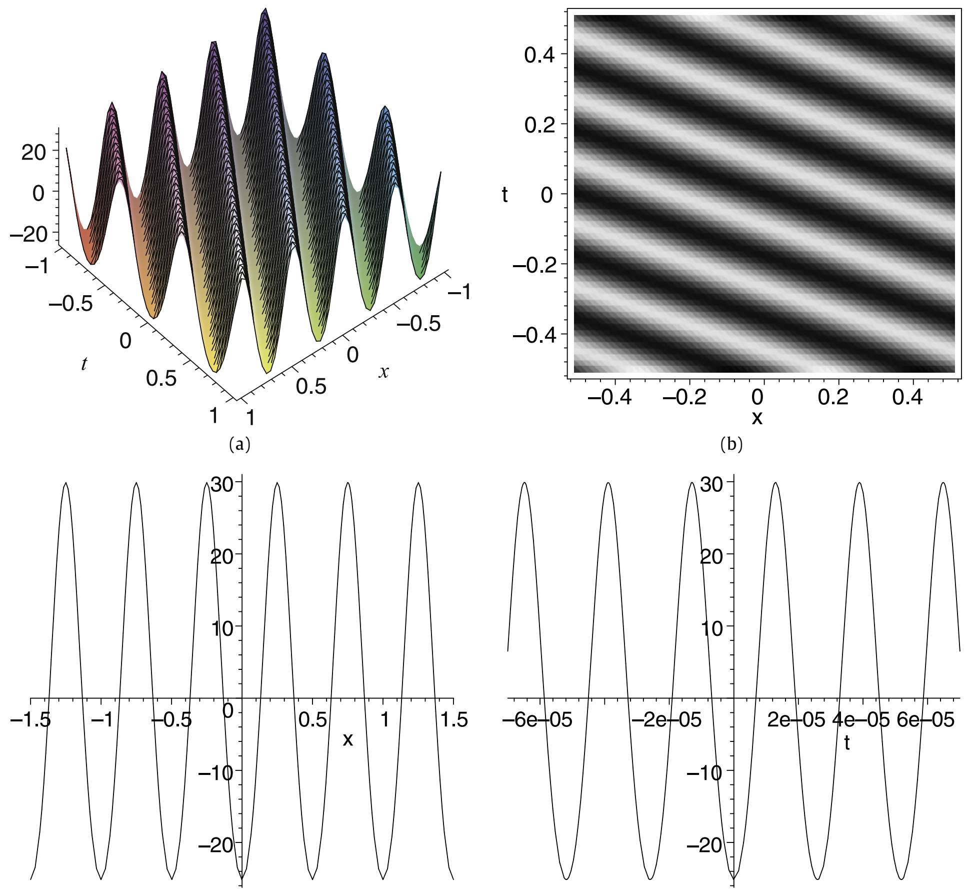
### 4.1. Construct one-periodic waves of the CDGSK equation

In this section, we consider one-periodic wave solutions of Eq. (4.1). According to Theorem 1, $k$ and $\omega$ should satisfy the following system

$$\sum_{n=-\infty}^{\infty}\big(-16\pi^2n^2k\omega+7680u_0\pi^4n^4k^4-4096\pi^6n^6k^6+c\big)e^{2n^2\pi i\tau}=0, \tag{4.5a}$$
$$\sum_{n=-\infty}^{\infty}\big(-4\pi^2(2n-1)^2k\omega+480u_0\pi^4(2n-1)^4k^4-64\pi^6(2n-1)^6k^6+c\big)e^{(2n^2-2n+1)\pi i\tau}=0. \tag{4.5b}$$

By introducing the notations as

$$a_{11}=-\sum_{n=-\infty}^{\infty}16\pi^2n^2k\wp^{2n^2},\qquad a_{12}=\sum_{n=-\infty}^{\infty}\wp^{2n^2},\qquad a_{21}=-\sum_{n=-\infty}^{\infty}4\pi^2(2n-1)^2k\wp^{2n^2-2n+1},$$
$$a_{22}=\sum_{n=-\infty}^{\infty}\wp^{2n^2-2n+1},\qquad b_1=-\sum_{n=-\infty}^{\infty}\big(7680u_0\pi^4n^4k^4-4096\pi^6n^6k^6\big)\wp^{2n^2},$$




(a) (b) (c) (d)

**Fig. 1.** (Color online) A one-periodic wave of the CDGSK equation with parameters: $u_0 = \varepsilon = 0$, $k = 2$, $\tau = i$. This figure shows that every one-periodic wave is one-dimensional, and it can be viewed as a superposition of overlapping solitary waves, placed one period apart. (a) Perspective view of the wave. (b) Overhead view of the wave, with contour plot shown. The bright lines are crests and the dark lines are troughs. (c) Wave propagation pattern of the wave along the $x$ axis. (d) Wave propagation pattern of the wave along the $t$ axis.

$$b_2 = -\sum_{n=-\infty}^{\infty} \left(480u_0\pi^4(2n-1)^4k^4 - 64\pi^6(2n-1)^6k^6\right)\wp^{2n^2-2n+1}, \quad \wp = e^{\pi i\tau}, \tag{4.6}$$

we simply change Eqs. (4.5) into a linear system about the frequency $\omega$ and the integration constant $c$, namely,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} \omega \\ c \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \tag{4.7}$$

Now solving this system, we get a one-periodic wave solution of Eq. (4.1)

$$u = u_0 + 2\partial_x^2 \ln\vartheta(\xi), \tag{4.8}$$

which provided the vector $(\omega, c)^T$ solves Eq. (4.7) with the theta function $\vartheta(\xi)$ given by Eq. (3.1) and parameters $\omega$, $c$ by Eq. (4.7). The other parameters $k$, $\tau$, $\varepsilon$ and $u_0$ are free. The two parameters $k$ and $\tau$ completely dominate a one-periodic wave. Fig. 1 shows a one-periodic wave for one choice of the parameters.

### 4.2. Feature and asymptotic property of one-periodic waves

The one-periodic wave solution (4.8) has a simple characterization as follows.

(i) It has two fundamental periods 1 and $\tau$ in the phase variable $\xi$.
(ii) It is actually a kind of one-dimensional cnoidal waves, i.e., there is a single phase variable $\xi$. Its speed parameter is given by

$$\omega = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}}. \tag{4.9}$$

(iii) It has only one wave pattern for all time and it can be viewed as a parallel superposition of overlapping one-solitary waves, placed one period apart (see Fig. 1).

Now we further consider asymptotic properties of the one-periodic wave solution. For this purpose, we have to use the solutions of the system (4.7). Since both the coefficient matrix and the right-side vector of system (4.7) are power series about $\wp$, its solution $(\omega, c)^T$ also should be a series about $\wp$. We can solve system (4.7) via small parameter expansion method and general procedure is described as follows.

We write the system (4.7) into power series of

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = A_0 + A_1\wp + A_2\wp^2 + \cdots, \tag{4.10}$$

$$\begin{pmatrix} \omega \\ c \end{pmatrix} = X_0 + X_1\wp + X_2\wp^2 + \cdots, \tag{4.11}$$

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = B_0 + B_1\wp + B_2\wp^2 + \cdots. \tag{4.12}$$

Substituting Eqs. (4.10)–(4.12) into Eq. (4.7) leads to the following recursion relations

$$A_0X_0 = B_0, \qquad A_0X_n + A_1X_{n-1} + \cdots + A_nX_0 = B_n, \quad n \geqslant 1,\ n \in \mathbb{N}, \tag{4.13}$$

form which we then recursively get each vector $X_i$, $i = 0, 1, \ldots.$

**Proposition 3.** *Assuming that the matrix $A_0$ is reversible, we can obtain*

$$X_0 = A_0^{-1}B_0, \qquad X_n = A_0^{-1}\left(B_n - \sum_{i=1}^{n} A_iB_{n-1}\right), \quad n \geqslant 1,\ n \in \mathbb{N}. \tag{4.14}$$

*If the matrix $A_0$ and $A_1$ are not inverse,*

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & 0 \\ -8\pi^2k & 2 \end{pmatrix},$$

*we can obtain*

$$X_0 = \begin{pmatrix} \frac{2B_0^{(1)} - B_1^{(2)}}{8\pi^2k} & B_0^{(1)} \end{pmatrix}^T, \qquad X_1 = \begin{pmatrix} \frac{2B_1^{(1)} - (B_2 - A_2X_0)^{(2)}}{8\pi^2k} & B_1^{(1)} \end{pmatrix}^T, \qquad \ldots,$$
$$X_n = \begin{pmatrix} \frac{2(B_{n+1} - \sum_{i=2}^{n} A_iX_{n-i})^{(1)} - (B_{n+1} - \sum_{i=2}^{n+1} A_iX_{n+1-i})^{(2)}}{8\pi^2k} & (B_{n+1} - \sum_{i=2}^{n} A_iX_{n-i})^{(1)} \end{pmatrix}^T, \quad n \geqslant 2,\ n \in \mathbb{N}, \tag{4.15}$$

*where $\alpha^{(1)}$ and $\alpha^{(2)}$ denote the first and second component of a two-dimensional vector $\alpha$, respectively.*

Interestingly, the relation between the one-periodic wave solution (4.8) and the one-soliton solution (4.4a) can be established as follows.

**Theorem 3.** *If the vector $(\omega, c)^T$ is a solution of the system* (4.7), *and for the one-periodic wave solution* (4.8), *we let*

$$u_0 = 0, \qquad k = \frac{\mu}{2\pi i}, \qquad \varepsilon = \frac{\delta + \pi\tau}{2\pi i}, \tag{4.16}$$

*where $\mu$ and $\delta$ are given in Eq.* (4.4a). *Then we have the following asymptotic properties*

$$c \to 0, \qquad \xi \to \frac{\eta + \pi\tau}{2\pi i}, \qquad \vartheta(\xi, \tau) \to 1 + e^{\eta}, \quad \text{when } \wp \to 0. \tag{4.17}$$

*It implies that the one-periodic solution* (4.8) *tends to the one-soliton solution* (4.4a) *under a small amplitude limit, that is $(u, \wp) \to (u_1, 0)$.*

**Proof.** By using Eqs. (4.6), we write functions $a_{ij}$, $b_i$, $i, j = 1, 2$, as the series about $\wp$

$$a_{11} = -32\pi^2\big(\wp^2 + 4\wp^8 + \cdots + n^2\wp^{2n^2} + \cdots\big), \qquad a_{12} = 1 + 2\big(\wp^2 + \wp^8 + \cdots + \wp^{2n^2} + \cdots\big),$$
$$a_{21} = -8\pi^2k\big(\wp + 9\wp^5 + \cdots + (2n-1)^2\wp^{2n^2-2n+1} + \cdots\big), \qquad a_{22} = 2\big(\wp + \wp^5 + \cdots + \wp^{2n^2-2n+1} + \cdots\big),$$
$$b_1 = 8192\pi^6k^6\big[\wp^2 + 64\wp^8 + \cdots + n^6\wp^{2n^2} + \cdots\big],$$
$$b_2 = 64\pi^6k^6\big[2\wp + 1458\wp^5 + \cdots + (2n-1)^6\wp^{2n^2-2n+1} + \cdots\big]. \tag{4.18}$$

With Eqs. (4.10) and (4.12), we have

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ -8\pi^2 k & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -32\pi^2 k & 2 \\ 0 & 0 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 0 & 0 \\ -72\pi^2 k & 2 \end{pmatrix}, \quad A_3 = A_4 = 0, \quad \ldots,$$
$$B_1 = \begin{pmatrix} 0 \\ 128\pi^6 k^6 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 8192\pi^6 k^6 \\ 0 \end{pmatrix}, \quad B_5 = \begin{pmatrix} 0 \\ 93312\pi^6 k^6 \end{pmatrix}, \quad B_0 = B_3 = B_4 = 0, \quad \ldots. \tag{4.19}$$

Substituting Eq. (4.19) into formulas (4.15), we can obtain

$$X_0 = \begin{pmatrix} -16\pi^4 k^5 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} -128\pi^4 k^5 \\ -512\pi^6 k^6 \end{pmatrix}, \quad X_4 = \begin{pmatrix} -10752\pi^4 k^5 \\ 3072\pi^6 k^6 \end{pmatrix}, \quad X_1 = X_3 = 0, \quad \ldots. \tag{4.20}$$

From (4.11), we then have

$$\begin{aligned} &\omega = -16\pi^4 k^5 - 128\pi^4 k^5 \wp^2 - 10752\pi^4 k^5 \wp^4 + o(\wp^4), \\ &c = -512\pi^6 k^6 \wp^2 + 3072\pi^6 k^6 \wp^4 + o(\wp^4), \end{aligned} \tag{4.21}$$

which implies by using relation (4.16) that

$$c \to 0, \qquad 2\pi i\omega \to -\mu^5, \quad \text{when } \wp \to 0. \tag{4.22}$$

To show the one-periodic wave (4.8) degenerates to the one-soliton solution (4.4a) under the limit $\wp \to 0$, we first expand the periodic function $\vartheta(\xi)$ in the form

$$\vartheta(\xi, \tau) = 1 + \left(e^{2\pi i\xi} + e^{-2\pi i\xi}\right)\wp + \left(e^{4\pi i\xi} + e^{-4\pi i\xi}\right)\wp^4 + \cdots. \tag{4.23}$$

By using the transformation (4.16), we have

$$\begin{aligned} &\vartheta(\xi, \tau) = 1 + e^{\tilde{\xi}} + \left(e^{-\tilde{\xi}} + e^{2\tilde{\xi}}\right)\wp^2 + \left(e^{-2\tilde{\xi}} + e^{3\tilde{\xi}}\right)\wp^6 + \cdots \to 1 + e^{\tilde{\xi}}, \quad \text{when } \wp \to 0, \\ &\tilde{\xi} = 2\pi i\xi - \pi\tau = \mu x + 2\pi i\omega t + \delta. \end{aligned} \tag{4.24}$$

Combining Eqs. (4.22) and (4.24) deduces that

$$\begin{aligned} &\tilde{\xi} \to \mu x - \mu^5 t + \delta = \eta, \quad \text{when } \wp \to 0, \\ &\xi \to \frac{\eta + \pi\tau}{2\pi i}, \quad \text{when } \wp \to 0. \end{aligned} \tag{4.25}$$

With Eqs. (4.24) and (4.25), we can obtain

$$\vartheta(\xi) \to 1 + e^{\eta}, \quad \text{when } \wp \to 0. \tag{4.26}$$

From above, we conclude that the one-periodic solution (4.8) just goes to the one-soliton solution (4.4a) as the amplitude $\wp \to 0$. □

### *4.3. Construct two-periodic waves of the CDGSK equation*

In this section, we consider two-periodic wave solutions of Eq. (4.2). According to Theorem 2, $k_i$ and $\omega_i$ should satisfy the following system

$$\begin{aligned} &\sum_{n\in\mathbb{Z}^2} \left[-4\pi^2\langle 2n-\theta_1, k\rangle\langle 2n-\theta_1, \omega\rangle + 480u_0\pi^4\langle 2n-\theta_1, k\rangle^4 - 64\pi^6\langle 2n-\theta_1, k\rangle^6 + c\rangle\right] \\ &\quad \times e^{\pi i[\langle\tau(n-\theta_1), n-\theta_1\rangle + \langle\tau n, n\rangle]} = 0, \end{aligned} \tag{4.27a}$$

$$\begin{aligned} &\sum_{n\in\mathbb{Z}^2} \left[-4\pi^2\langle 2n-\theta_2, k\rangle\langle 2n-\theta_2, \omega\rangle + 480u_0\pi^4\langle 2n-\theta_2, k\rangle^4 - 64\pi^6\langle 2n-\theta_2, k\rangle^6 + c\rangle\right] \\ &\quad \times e^{\pi i[\langle\tau(n-\theta_2), n-\theta_2\rangle + \langle\tau n, n\rangle]} = 0, \end{aligned} \tag{4.27b}$$

$$\begin{aligned} &\sum_{n\in\mathbb{Z}^2} \left[-4\pi^2\langle 2n-\theta_3, k\rangle\langle 2n-\theta_3, \omega\rangle + 480u_0\pi^4\langle 2n-\theta_3, k\rangle^4 - 64\pi^6\langle 2n-\theta_3, k\rangle^6 + c\rangle\right] \\ &\quad \times e^{\pi i[\langle\tau(n-\theta_3), n-\theta_3\rangle + \langle\tau n, n\rangle]} = 0, \end{aligned} \tag{4.27c}$$

$$\begin{aligned} &\sum_{n\in\mathbb{Z}^2} \left[-4\pi^2\langle 2n-\theta_4, k\rangle\langle 2n-\theta_4, \omega\rangle + 480u_0\pi^4\langle 2n-\theta_4, k\rangle^4 - 64\pi^6\langle 2n-\theta_4, k\rangle^6 + c\rangle\right] \\ &\quad \times e^{\pi i[\langle\tau(n-\theta_4), n-\theta_4\rangle + \langle\tau n, n\rangle]} = 0. \end{aligned} \tag{4.27d}$$

By introducing the notations as

$$
\begin{aligned}
&H=(h_{ij})_{4\times 4}, \qquad b=(b_1,b_2,b_3,b_4)^T,\\
&h_{i1}=-4\pi^2\sum_{(n_1,n_2)\in\mathbb{Z}^2}\langle 2n-\theta_i,k\rangle\big(2n_1-\theta_i^1\big)\Im_i(n), \qquad h_{i2}=-4\pi^2\sum_{(n_1,n_2)\in\mathbb{Z}^2}\langle 2n-\theta_i,k\rangle\big(2n_2-\theta_i^2\big)\Im_i(n),\\
&h_{i3}=480\pi^4\sum_{(n_1,n_2)\in\mathbb{Z}^2}\langle 2n-\theta_i,k\rangle^4\Im_i(n), \qquad h_{i4}=\sum_{(n_1,n_2)\in\mathbb{Z}^2}\Im_i(n), \qquad b_i=64\pi^6\sum_{(n_1,n_2)\in\mathbb{Z}^2}\langle 2n-\theta_i,k\rangle^6\Im_i(n),\\
&\Im_i(n)=\wp_1^{n_1^2+(n_1-\theta_i^1)^2}\wp_2^{n_2^2+(n_2-\theta_i^2)^2}\wp_3^{n_1n_2+(n_1-\theta_i^1)(n_2-\theta_i^2)},\\
&\wp_1=e^{\pi i\tau_{11}}, \qquad \wp_2=e^{\pi i\tau_{22}}, \qquad \wp_3=e^{2\pi i\tau_{12}}, \quad i=1,2,3,4,
\end{aligned}
\tag{4.28}
$$

Eqs. (4.27) can be written as a linear system

$$H(\omega_1,\omega_2,u_0,c)^T=b. \tag{4.29}$$

Now solving this system, we get a two-periodic wave solution of Eq. (4.1)

$$u=u_0+2\partial_x^2\ln\vartheta(\xi_1,\xi_2,\tau), \tag{4.30}$$

where $\vartheta(\xi_1,\xi_2,\tau)$ and parameters $\omega_1$, $\omega_2$, $u_0$, $c$ are given by Eqs. (3.7) and (4.29), respectively. The other parameters $k_1$, $k_2$, $\varepsilon_1$, $\varepsilon_2$, $\tau_{11}$, $\tau_{12}$ and $\tau_{22}$ are free. The two-periodic wave is specified by five of the parameters $k_1$, $k_2$, $\tau_{11}$, $\tau_{12}$ and $\tau_{22}$.

### *4.4. Feature and asymptotic property of two-periodic waves*

The two-periodic wave solution (4.30) has a simple characterization as follows.

(i) It is a direct generalization of one-periodic waves, its surface pattern is two-dimensional, i.e., there are two phase variables $\xi_1$ and $\xi_2$, which has two independent spatial periods in two independent horizontal directions.
(ii) It has $2N$ fundamental periods $\{\zeta_i,\ i=1,2,\ldots,N\}$ and $\{\tau_i,\ i=1,2,\ldots,N\}$ in $(\xi_1,\xi_2)$ with $\zeta_1=(1,0,\ldots,0)^T,\ldots,\zeta_N=(0,0,\ldots,1)^T$.
(iii) Assuming that parameters satisfy a ratio relation

$$\omega_2\sim d\omega_1, \qquad \xi_2\sim d\xi_1, \qquad \vartheta(\xi_1,\xi_2)\sim\vartheta(\xi_1,d\xi_1), \tag{4.31}$$

the two-periodic wave is actually one-dimensional and it degenerates to one-periodic wave (see Fig. 2).
(iv) If parameters do not satisfy the ratio relation (4.31), then for any time $t$, phase variables $\xi_1=d_1$ and $\xi_2=d_2$ ($d_1$, $d_2$ are constants) intersect at a unique point. This point moves in the $(x,t)$-plane with a constant speed as the time $t$ changes. Every two-periodic wave as in Fig. 3 is spatially periodic in two directions, but it need not be periodic in either $x$ or $t$ directions. The basic cell of the pattern seems like a regular quadrilateral: four steep wave crests form the edges of each quadrilateral.
(v) In a subcase of the (iv) $\tau_{11}=\tau_{12}$, the two-periodic solution has only four independent parameters $k_1$, $k_2$, $\tau_{11}$ and $\tau_{12}$, which is called a symmetric solution [42]. The graph of $u$ is in Fig. 4. It is shown that the cell of its pattern is a regular quadrilateral from the contour plot (see Fig. 4(b)).

Finally, we consider the asymptotic properties of the two-periodic solution (4.30). In a way similar to Theorem 3, we can establish the relation between the two-periodic solution (4.30) and the two-soliton solution (4.4b) as follows.

**Theorem 4.** *If $(\omega_1,\omega_2,u_0,c)^T$ is a solution of the system* (4.29)*, and for the two-periodic wave solution* (4.30)*, we take*

$$k_i=\frac{\mu_i}{2\pi i}, \qquad \varepsilon_i=\frac{\delta_i+\pi\tau_{ij}}{2\pi i}, \qquad \tau_{12}=\frac{A_{12}}{2\pi i}, \quad i=1,2, \tag{4.32}$$

*where $\mu_i$, $\delta_i$, $i=1,2$, and $A_{12}$ are given in Eq.* (4.4b)*. Then we have the following asymptotic relations*

$$
\begin{aligned}
&u_0\to 0, \qquad c\to 0, \qquad \xi_i\to\frac{\eta_i+\pi\tau_{ii}}{2\pi i}, \quad i=1,2,\\
&\vartheta(\xi_1,\xi_2,\tau)\to 1+e^{\eta_1}+e^{\eta_2}+e^{\eta_1+\eta_1+A_{12}}, \quad \text{as } \wp_1,\wp_2\to 0.
\end{aligned}
\tag{4.33}
$$

*It implies that the two-periodic solution* (4.30) *tends to the two-soliton solution* (4.4b) *under a small amplitude limit, that is* $(u,\wp_1,\wp_2)\to(u_1,0,0)$.

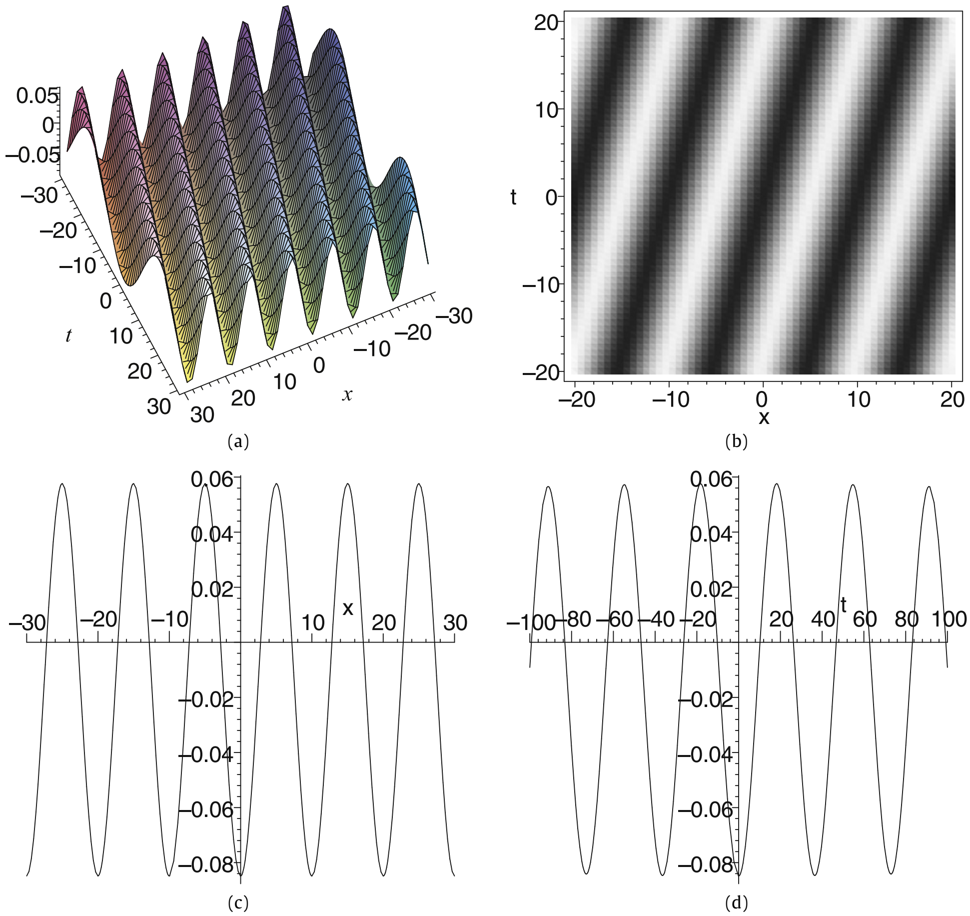



**Fig. 2.** (Color online) A degenerate two-periodic wave solution to the CDGSK equation with parameters: $\frac{k_1}{k_2}=\frac{\tau_{11}}{\tau_{22}}$ and $k_1=0.1$, $k_2=0.3$, $\tau_{11}=i$, $\tau_{12}=0.5i$, $\tau_{22}=3i$, $\varepsilon_1=\varepsilon_2=0$. This figure shows that the degenerate two-periodic wave is almost one-dimensional. (a) Perspective view of the wave. (b) Overhead view of the wave, with contour plot shown. The bright lines are crests and the dark lines are troughs. (c) Wave propagation pattern of the wave along the $x$ axis. (d) Wave propagation pattern of wave along the $t$ axis.

**Proof.** The periodic wave function $\vartheta(\xi_1,\xi_2,\tau)$ is expand in the following form

$$\begin{aligned}\vartheta(\xi_1,\xi_2,\tau)=1&+\left(e^{2\pi i\xi_1}+e^{-2\pi i\xi_1}\right)e^{\pi\tau_{11}}+\left(e^{2\pi i\xi_2}+e^{-2\pi i\xi_2}\right)e^{\pi\tau_{22}}\\&+\left(e^{2\pi i(\xi_1+\xi_2)}+e^{-2\pi i(\xi_1+\xi_2)}\right)e^{\pi(\tau_{11}+2\tau_{12}+\tau_{22})}+\cdots.\end{aligned}\tag{4.34}$$

By using Eq. (4.32), we get

$$\begin{aligned}\vartheta(\xi_1,\xi_2,\tau)&=1+e^{\tilde\xi_1}+e^{\tilde\xi_2}+e^{\tilde\xi_1+\tilde\xi_2-2\pi\tau_{12}}+\wp_1^2e^{-\tilde\xi_1}+\wp_2^2e^{-\tilde\xi_2}+\wp_1^2\wp_2^2e^{-\tilde\xi_1^2-\tilde\xi_2-2\pi\tau_{12}}+\cdots\\&\to 1+e^{\tilde\xi_1}+e^{\tilde\xi_2}+e^{\tilde\xi_1+\tilde\xi_2+A_{12}},\quad \text{as } \wp_1,\wp\to 0,\end{aligned}\tag{4.35}$$

where $\tilde\xi_i=\mu_ix-\mu_i^5t+\delta_i$, $\tilde\omega_i=2\pi i\omega_i$, $i=1,2$.

It remains to prove that

$$c\to 0,\qquad \tilde\omega_i\to-\mu_i^5,\qquad \tilde\xi_i\to\eta_i,\quad i=1,2,\ \text{as } \wp_1,\wp_2\to 0.\tag{4.36}$$

Similar to Eq. (4.18), we can expand each function in $\{h_{ij},\ b_i,\ i=1,2,3,4\}$ into a series with $\wp_1$, $\wp_2$. In fact, we only need to make the first order expansions of matrix $H$ and vector $b$ with $\wp_1$, $\wp_2$ to show the asymptotic relations (4.36). Now we consider their second order expansions to see deeper relations among parameters for the two-periodic solution (4.30) and the two-soliton solution (4.4b). The expansions for the matrix $H$, the vector $b$ and the solution of the system (4.29) are given by

$$H=\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}+\begin{pmatrix}0&0&0&0\\-8\pi^2k_1&0&960\pi^4k_1^4&2\\0&0&0&0\\0&0&0&0\end{pmatrix}\wp_1+\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&-8\pi^2k_2&960\pi^4k_2^4&2\\0&0&0&0\end{pmatrix}\wp_2$$

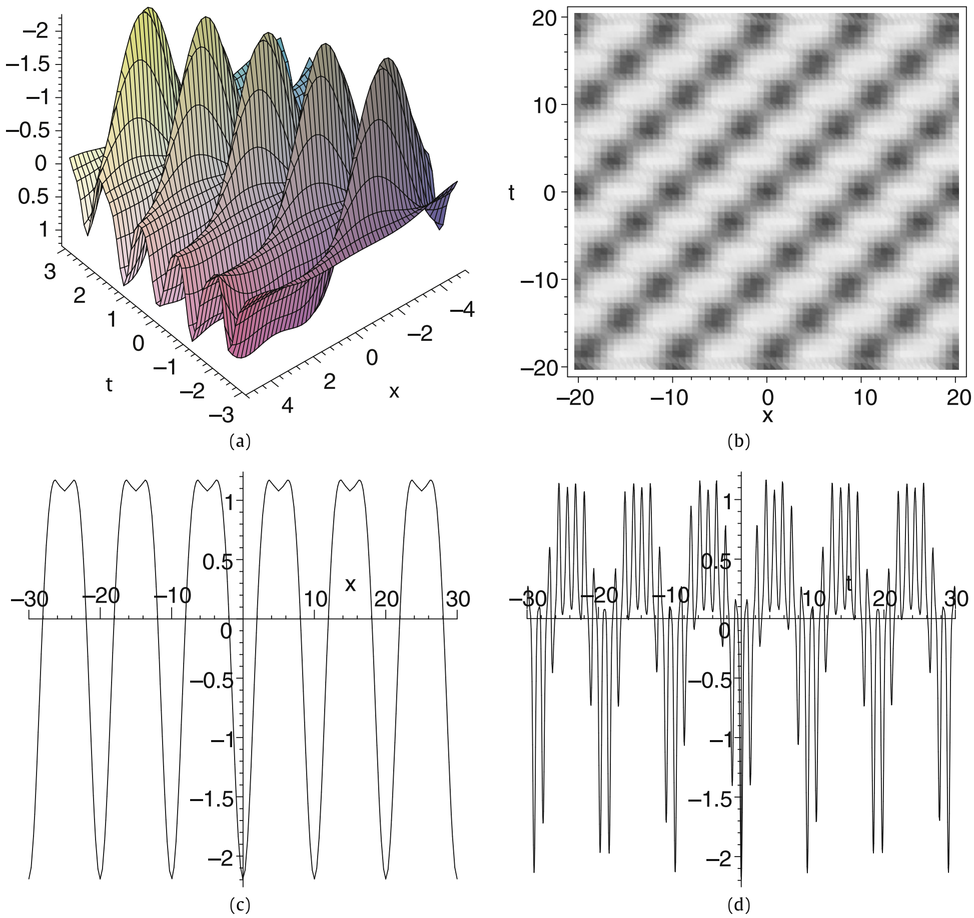



**Fig. 3.** (Color online) An asymmetric two-periodic breather type wave to the CDGSK equation with parameters: $\frac{k_1}{k_2} \neq \frac{\tau_{11}}{\tau_{22}}$ and $k_1 = 0.1$, $k_2 = 0.2$, $\tau_{11} = 0.3i$, $\tau_{12} = 0.5i$, $\tau_{22} = i$, $\varepsilon_1 = \varepsilon_2 = 0$. This figure shows that the asymmetric two-periodic wave is spatially periodic in two directions, but it need not be periodic in either the $x$ or $t$ directions. (a) Perspective view of the wave. (b) Overhead view of the wave, with contour plot shown. The bright quadrilaterals are crests and the dark quadrilaterals are troughs. (c) Wave propagation pattern of the wave along the $x$ axis. (d) Wave propagation pattern of wave along the $t$ axis.

$$
\begin{aligned}
&+\begin{pmatrix} -32\pi^2 k_1 & 0 & 15360\pi^4 k_1^4 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \wp_1^2 + \begin{pmatrix} 0 & -32\pi^2 k_2 & 15360\pi^4 k_2^4 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \wp_2^2 \\
&+\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Delta_1 & \Delta_2 & \Delta_3 & \Delta_4 \end{pmatrix} \wp_1 \wp_2 + o\big(\wp_1^i \wp_2^j\big), \quad i + j \geqslant 2,
\end{aligned} \tag{4.37}
$$

$$
\begin{aligned}
b = {}& \begin{pmatrix} 0 \\ 128\pi^6 k_1^6 \\ 0 \\ 0 \end{pmatrix} \wp_1 + \begin{pmatrix} 0 \\ 0 \\ 128\pi^6 k_2^6 \\ 0 \end{pmatrix} \wp_2 + \begin{pmatrix} 8192\pi^6 k_1^6 \\ 0 \\ 0 \\ 0 \end{pmatrix} \wp_1^2 + \begin{pmatrix} 8192\pi^6 k_2^6 \\ 0 \\ 0 \\ 0 \end{pmatrix} \wp_2^2 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \Gamma \end{pmatrix} \wp_1 \wp_2 \\
&+ o\big(\wp_1^i \wp_2^j\big), \quad i + j \geqslant 2,
\end{aligned} \tag{4.38}
$$

$$
\begin{aligned}
\begin{pmatrix} \omega_1 \\ \omega_2 \\ u_0 \\ c \end{pmatrix} = {}& \begin{pmatrix} \omega_1^{(00)} \\ \omega_2^{(00)} \\ u_0^{(00)} \\ c^{(00)} \end{pmatrix} + \begin{pmatrix} \omega_1^{(11)} \\ \omega_2^{(11)} \\ u_0^{(11)} \\ c^{(11)} \end{pmatrix} \wp_1 + \begin{pmatrix} \omega_1^{(21)} \\ \omega_2^{(21)} \\ u_0^{(21)} \\ c^{(21)} \end{pmatrix} \wp_2 + \begin{pmatrix} \omega_1^{(12)} \\ \omega_2^{(12)} \\ u_0^{(12)} \\ c^{(12)} \end{pmatrix} \wp_1^2 + \begin{pmatrix} \omega_1^{(22)} \\ \omega_2^{(22)} \\ u_0^{(22)} \\ c^{(22)} \end{pmatrix} \wp_2^2 + \begin{pmatrix} \omega_1^{(2)} \\ \omega_2^{(2)} \\ u_0^{(2)} \\ c^{(2)} \end{pmatrix} \wp_1 \wp_2 \\
&+ o\big(\wp_1^i \wp_2^j\big), \quad i + j \geqslant 2,
\end{aligned} \tag{4.39}
$$

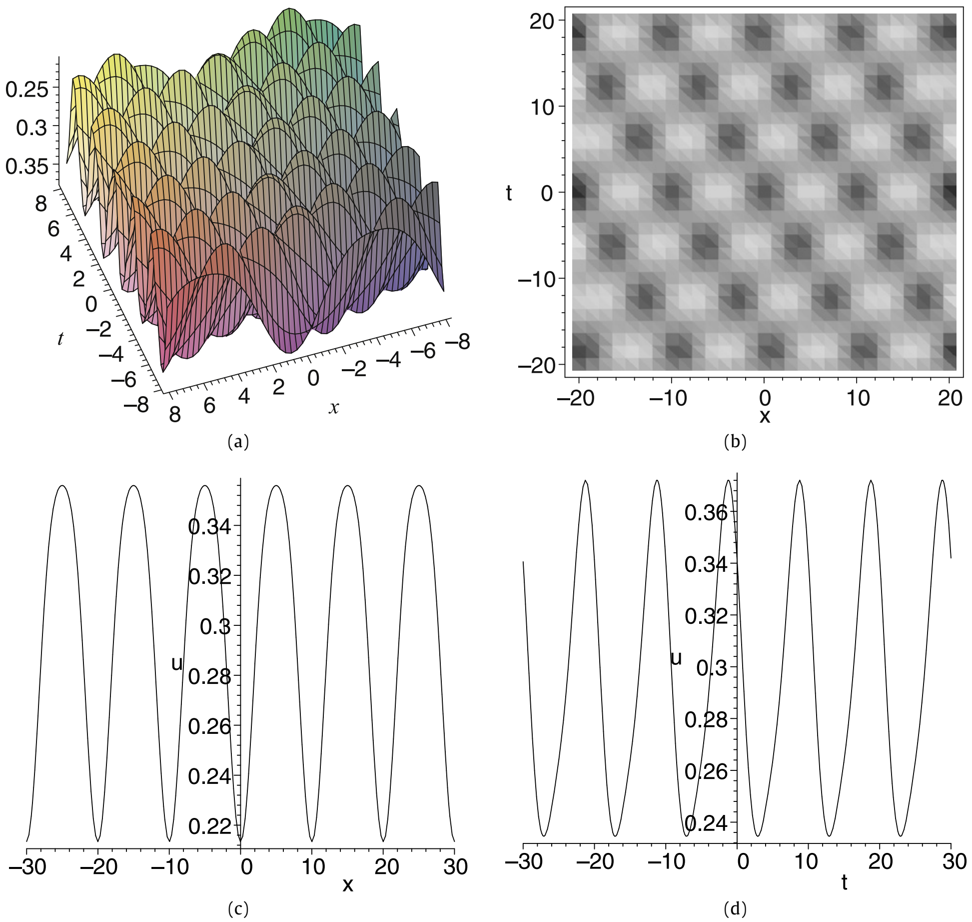



**Fig. 4.** (Color online) A symmetric two-periodic breather type wave for the CDGSK equation with parameters: $\frac{k_1}{k_2} \neq \frac{\tau_{11}}{\tau_{22}}$ and $k_1 = 0.1$, $k_2 = 0.2$, $\tau_{11} = i$, $\tau_{12} = i$, $\tau_{22} = 2i$, $\varepsilon_1 = \varepsilon_2 = 0$. This figure shows that the symmetric two-periodic wave is periodic in two directions. (a) Perspective view of the wave. (b) Overhead view of the wave, with contour plot shown. The bright quadrilaterals are crests and the dark quadrilaterals are troughs. (c) Wave propagation pattern of the wave along the $x$ axis. (d) Wave propagation pattern of wave along the $t$ axis.

where $\Delta_1 = -8\pi^2(k_1-k_2) - 8\pi^2(k_1+k_2)\wp_3$, $\Delta_2 = 8\pi^2(k_1-k_2) - 8\pi^2(k_1+k_2)\wp_3$, $\Delta_3 = 960\pi^4(k_1-k_2)^4 + 960\pi^4(k_1+k_2)^4\wp_3$, $\Delta_4 = 2 + 2\wp_3$, $\Gamma = 128\pi^6(k_1-k_2)^6 + 128\pi^6(k_1+k_2)^6\wp_3$.

Substituting Eqs. (4.37)–(4.39) into system (4.29) and comparing the same order of $\wp_1$ and $\wp_2$, we obtain the following relations

$$\begin{aligned}
&c^{(00)} = c^{(11)} = c^{(21)} = c^{(2)} = 0,\\
&\omega_1^{(11)} - 120\pi^2 k_1^3 u_0^{(11)} = 0, \qquad \omega_2^{(11)} - 120\pi^2 k_2^3 u_0^{(11)} = 0, \qquad \omega_2^{(21)} - 120\pi^2 k_2^3 u_0^{(21)} = 0,\\
&\omega_1^{(21)} - 120\pi^2 k_1^3 u_0^{(21)} = 0, \qquad \omega_1^{(00)} - 120\pi^2 k_1^3 u_0^{(00)} = -16\pi^4 k_1^5, \qquad \omega_2^{(00)} - 120\pi^2 k_2^3 u_0^{(00)} = -16\pi^4 k_2^5,\\
&c^{(12)} - 32\pi^2 k_1 \omega_1^{(00)} + 15360\pi^4 k_1^4 u_0^{(00)} = 8192\pi^6 k_1^6,\\
&(k_1-k_2)\omega_1^{(00)} + (k_1+k_2)\wp_3\omega_1^{(00)} - (k_1-k_2)\omega_2^{(00)} + (k_1+k_2)\wp_3\omega_2^{(00)} - 120\pi^2(k_1-k_2)^4 u_0^{(00)}\\
&\quad - 120\pi^2(k_1+k_2)^4\wp_3 u_0^{(00)} = -16\pi^4(k_1-k_2)^5 - 16\pi^4(k_1+k_2)^5\wp_3,\\
&c^{(22)} - 32\pi^2 k_2 \omega_2^{(00)} + 15360\pi^4 k_2^4 u_0^{(00)} = 8192\pi^6 k_2^6.
\end{aligned} \tag{4.40}$$

Choosing $u_0^{(00)} = 0$, we can obtain

$$\begin{aligned}
&u_0 = o(\wp_1, \wp_2) \to 0, \qquad c = 7680\pi^6 k_1^6 \omega\wp_1^2 + 7680\pi^6 k_2^6 \omega\wp_2^2 + o(\wp_1\wp_2) \to 0,\\
&\omega_1 = -16\pi^4 k_1^5 + o(\wp_1\wp_2) \to -16\pi^4 k_1^5,\\
&\omega_2 = -16\pi^4 k_2^5 + o(\wp_1\wp_2) \to -16\pi^4 k_2^5, \quad \text{as } (\wp_1, \wp_2) \to (0,0),
\end{aligned} \tag{4.41}$$

which implies Eq. (4.36). From above, we conclude that the two-periodic solution (4.30) tends to the two-soliton solution (4.4b) as $(\wp_1, \wp_2) \to (0, 0)$. □

## 5. The $(2+1)$-dimensional breaking soliton equation

We consider the $(2+1)$-dimensional breaking soliton equation ($(2+1)$DBS) [43]

$$u_t + u_{xxy} - 4uu_y - 2u_x v = 0, \qquad u_y = v_x, \tag{5.1}$$

which is related to the following breaking soliton equation

$$u_t + u_{xxy} - 4uu_y - 2u_x \partial_x^{-1} u_y = 0. \tag{5.2}$$

In this section we construct its periodic wave solution and show that the soliton solution can be obtained as limiting case of the periodic solution. Applying the Hirota–Riemann method to construct periodic wave solutions of Eq. (5.2), we consider a variable transformation

$$u = u_0 - 2\partial_x^2 \ln \vartheta(\xi), \tag{5.3}$$

where $\xi = kx + \rho y + \omega t + \varepsilon$. Integrating with respect to $x$, Eq. (2.10) becomes the following bilinear form

$$\mathscr{L}(D_x, D_y, D_t)\vartheta(\xi) \cdot \vartheta(\xi) = \big(D_t D_x - 4u_0 D_x D_y + D_y D_x^3 + c\big)\vartheta(\xi) \cdot \vartheta(\xi) = 0. \tag{5.4}$$

According to Eqs. (2.7)–(2.8), Eq. (5.2) admits a one-soliton solution and two-soliton solution

$$u_1 = 2\partial_x^2 \ln\big(1 + e^{\eta}\big), \tag{5.5a}$$

$$u_2 = 2\partial_x^2 \ln\big(1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1+\eta_2+A_{12}}\big), \tag{5.5b}$$

where phase variable $\eta = \mu x + \nu y - \mu^2 \nu t + \delta$, $\eta_i = \mu_i x + \nu_i y - \mu_i^2 \nu_i t + \delta_i$, $i = 1, 2$, $e^{A_{12}} = \frac{(\mu_1-\mu_2)(-\mu_1^2\nu_2 - 2\mu_1\mu_2\nu_1 + 2\mu_1\mu_2\nu_2 + \mu_2^2\nu_1)}{(\mu_1+\mu_2)(\mu_1^2\nu_2 + 2\mu_1\mu_2\nu_1 + 2\mu_1\mu_2\nu_2 + \mu_2^2\nu_1)}$, $\mu$, $\rho$, $\delta$, $\mu_i$, $\rho_i$, $\delta_i$, $i = 1, 2$, are constants.

### 5.1. *Construct one-periodic waves of the* $(2+1)$*DBS equation*

In this section, we consider one-periodic wave solutions of Eq. (5.2). According to Theorem 1, $k$, $\rho$ and $\omega$ should satisfy the following system

$$\sum_{n=-\infty}^{\infty} \big(-16\pi^2 n^2 k\omega + 64u_0\pi^2 n^2 k\rho + 256\pi^4 n^4 k^3 \rho + c\big)e^{2n^2\pi i\tau} = 0, \tag{5.6a}$$

$$\sum_{n=-\infty}^{\infty} \big(-4\pi^2(2n-1)^2 k\omega + 16u_0\pi^4(2n-1)^2 k\rho + 16\pi^4(2n-1)^4 k^3\rho + c\big)e^{(2n^2-2n+1)\pi i\tau} = 0. \tag{5.6b}$$

By introducing the notations as

$$a_{11} = -\sum_{n=-\infty}^{\infty} 16\pi^2 n^2 k \wp^{2n^2}, \qquad a_{12} = \sum_{n=-\infty}^{\infty} \wp^{2n^2}, \qquad a_{21} = -\sum_{n=-\infty}^{\infty} 4\pi^2(2n-1)^2 k \wp^{2n^2-2n+1},$$

$$a_{22} = \sum_{n=-\infty}^{\infty} \wp^{2n^2-2n+1}, \qquad b_1 = -\sum_{n=-\infty}^{\infty} \big(64u_0\pi^2 n^2 k\rho + 256\pi^4 n^4 k^3 \rho\big)\wp^{2n^2},$$

$$b_2 = -\sum_{n=-\infty}^{\infty} \big(16u_0\pi^4(2n-1)^2 k\rho + 16\pi^4(2n-1)^4 k^3\rho\big)\wp^{2n^2-2n+1}, \quad \wp = e^{\pi i\tau}, \tag{5.7}$$

we change Eqs. (5.6) into the following linear system about the frequency $\omega$ and the integration constant $c$, namely,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \omega \\ c \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \tag{5.8}$$

Now solving this system, we get a one-periodic wave solution of Eq. (5.2)

$$u = u_0 - 2\partial_x^2 \ln \vartheta(\xi), \tag{5.9}$$

which provided the vector $(\omega, c)^T$ solves Eq. (5.8) with the theta function $\vartheta(\xi)$ given by Eq. (3.1) and parameters $\omega$, $c$ by Eq. (5.8). The other parameters $k$, $\rho$, $\tau$, $\varepsilon$ and $u_0$ are free. The three parameters $k$, $\rho$ and $\tau$ completely dominate a one-periodic wave. Fig. 5 shows a one-periodic wave for one choice of the parameters.

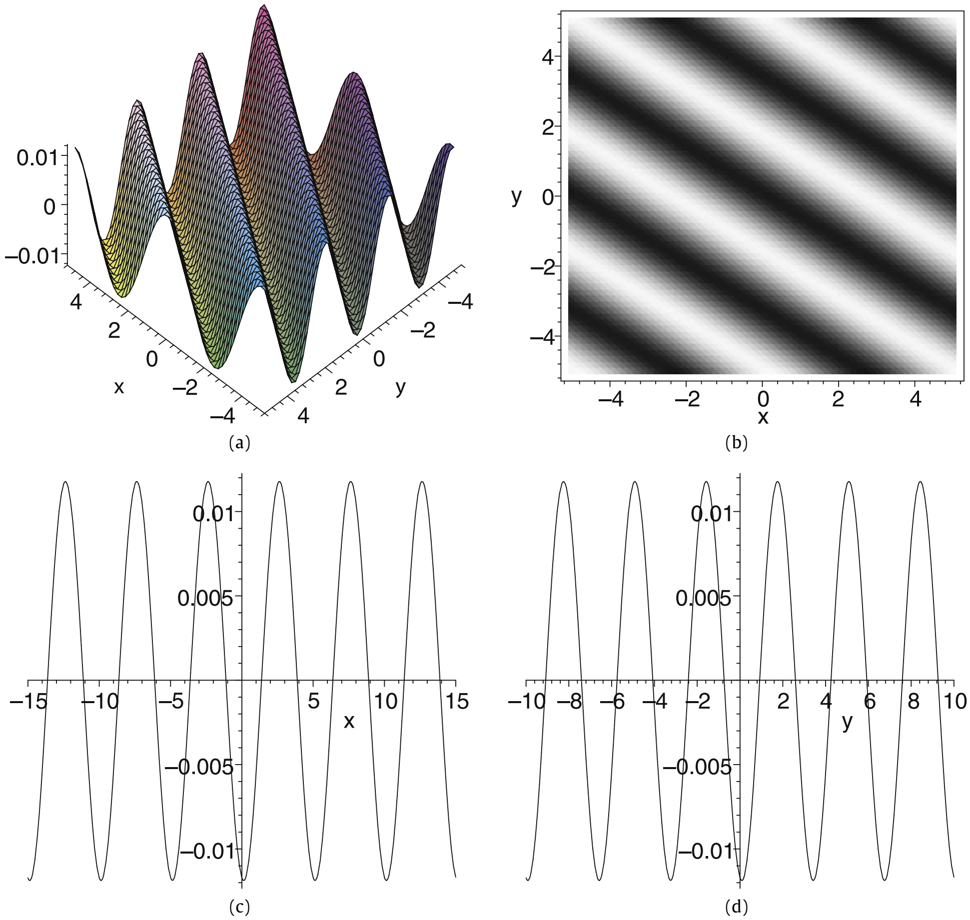



**Fig. 5.** (Color online) A one-periodic wave of the $(2+1)$DBS equation with parameters: $u_0 = \varepsilon = 0$, $k = 0.2$, $\rho = 0.3$, $\tau = 2i$. This figure shows that every one-periodic wave is one-dimensional, and it can be viewed as a superposition of overlapping solitary waves, placed one period apart. (a) Perspective view of the wave. (b) Overhead view of the wave, with contour plot shown. The bright lines are crests and the dark lines are troughs. (c) Wave propagation pattern of the wave along the $x$ axis. (d) Wave propagation pattern of the wave along the $y$ axis.

### *5.2. Feature and asymptotic property of one-periodic waves*

The one-periodic wave solution (5.9) has some simple characterizations that are similar to the CDGSK equation. In the following, we mainly consider the asymptotic property of one-periodic wave solutions. For this purpose, we have to use the solutions of the system (5.8). Since both the coefficient matrix and the right-side vector of system (5.8) are power series about $\wp$, its solution $(\omega, c)^T$ also should be a series about $\wp$. We can solve system (5.8) via small parameter expansion method and general procedure is similar to Eqs. (4.10)–(4.13) and Proposition 3.

Interestingly, the relation between the one-periodic wave solution (5.9) and the one-soliton solution (5.5a) can be established as follows.

**Theorem 5.** *If the vector* $(\omega, c)^T$ *is a solution of the system* (5.8)*, and for the one-periodic wave solution* (5.9)*, we let*

$$u_0 = 0, \qquad k = \frac{\mu}{2\pi i}, \qquad \rho = \frac{\nu}{2\pi i}, \qquad \varepsilon = \frac{\delta + \pi\tau}{2\pi i}, \tag{5.10}$$

*where* $\mu$*,* $\nu$ *and* $\delta$ *are given in Eq.* (5.5a)*. Then we have the following asymptotic properties*

$$c \to 0, \qquad \xi \to \frac{\eta + \pi\tau}{2\pi i}, \qquad \vartheta(\xi, \tau) \to 1 + e^{\eta}, \quad \text{when } \wp \to 0. \tag{5.11}$$

*It implies that the one-periodic solution* (5.9) *tends to the one-soliton solution* (5.5a) *under a small amplitude limit, that is* $(u, \wp) \to (u_1, 0)$.

**Proof.** With Eqs. (5.7), we write functions $a_{ij}$, $b_i$, $i, j = 1, 2$, as the series about $\wp$

$$\begin{aligned}
&a_{11} = -32\pi^2\big(\wp^2 + 4\wp^8 + \cdots + n^2\wp^{2n^2} + \cdots\big), \qquad a_{12} = 1 + 2\big(\wp^2 + \wp^8 + \cdots + \wp^{2n^2} + \cdots\big),\\
&a_{21} = -8\pi^2 k\big(\wp + 9\wp^5 + \cdots + (2n-1)^2\wp^{2n^2-2n+1} + \cdots\big), \qquad a_{22} = 2\big(\wp + \wp^5 + \cdots + \wp^{2n^2-2n+1} + \cdots\big),\\
&b_1 = -512\pi^4 k^3\rho\big[\wp^2 + 16\wp^8 + \cdots + n^4\wp^{2n^2} + \cdots\big],\\
&b_2 = -16\pi^4 k^3\rho\big[2\wp + 162\wp^5 + \cdots + 2(2n-1)^4\wp^{2n^2-2n+1} + \cdots\big].
\end{aligned} \tag{5.12}$$

By using Eqs. (4.10) and (4.12), we have

$$\begin{aligned}
&A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ -8\pi^2 k & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -32\pi^2 k & 2 \\ 0 & 0 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 0 & 0 \\ -72\pi^2 k & 2 \end{pmatrix}, \quad A_3 = A_4 = 0, \quad \ldots,\\
&B_1 = \begin{pmatrix} 0 \\ -32\pi^4 k^3\rho \end{pmatrix}, \quad B_2 = \begin{pmatrix} -512\pi^4 k^3\rho \\ 0 \end{pmatrix}, \quad B_5 = \begin{pmatrix} 0 \\ -2592\pi^4 k^3\rho \end{pmatrix}, \quad B_0 = B_3 = B_4 = 0, \quad \ldots.
\end{aligned} \tag{5.13}$$

Substituting Eq. (5.13) into formulas (4.15), we can obtain

$$X_0 = \begin{pmatrix} 4\pi^2 k^2\rho \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 32\pi^2 k^2\rho \\ 128\pi^4 k^3\rho \end{pmatrix}, \quad X_4 = \begin{pmatrix} 96\pi^2 k^2\rho \\ -768\pi^4 k^3\rho \end{pmatrix}, \quad X_1 = X_3 = 0, \quad \ldots. \tag{5.14}$$

From (4.11), we then have

$$\begin{aligned}
&\omega = 4\pi^2 k^2\rho + 32\pi^2 k^2\rho\wp^2 + 96\pi^2 k^2\rho\wp^4 + o\big(\wp^4\big),\\
&c = 128\pi^4 k^3\rho\wp^2 - 768\pi^4 k^3\rho\wp^4 + o\big(\wp^4\big),
\end{aligned} \tag{5.15}$$

which implies by using relation (5.10) that

$$c \to 0, \qquad 2\pi i\omega \to -\mu^2\nu, \quad \text{when } \wp \to 0. \tag{5.16}$$

To show the one-periodic wave (5.9) degenerates to the one-soliton solution (5.5a) under the limit $\wp \to 0$, we first expand the periodic function $\vartheta(\xi)$ in the form

$$\vartheta(\xi,\tau) = 1 + \big(e^{2\pi i\xi} + e^{-2\pi i\xi}\big)\wp + \big(e^{4\pi i\xi} + e^{-4\pi i\xi}\big)\wp^4 + \cdots. \tag{5.17}$$

By using the transformation (5.10), we have

$$\begin{aligned}
&\vartheta(\xi,\tau) = 1 + e^{\tilde\xi} + \big(e^{-\tilde\xi} + e^{2\tilde\xi}\big)\wp^2 + \big(e^{-2\tilde\xi} + e^{3\tilde\xi}\big)\wp^6 + \cdots \to 1 + e^{\tilde\xi}, \quad \text{when } \wp \to 0,\\
&\tilde\xi = 2\pi i\xi - \pi\tau = \mu x + \nu y + 2\pi i\omega t + \delta.
\end{aligned} \tag{5.18}$$

Combining Eqs. (5.16) and (5.18) deduces that

$$\begin{aligned}
&\tilde\xi \to \mu x + \nu y - \mu^2\nu t + \delta = \eta, \quad \text{when } \wp \to 0,\\
&\xi \to \frac{\eta + \pi\tau}{2\pi i}, \quad \text{when } \wp \to 0.
\end{aligned} \tag{5.19}$$

With Eqs. (5.18) and (5.19), we can obtain

$$\vartheta(\xi) \to 1 + e^{\eta}, \quad \text{when } \wp \to 0. \tag{5.20}$$

From above, we conclude that the one-periodic solution (5.9) just goes to the one-soliton solution (5.5a) as the amplitude $\wp \to 0$. □

### *5.3. Construct two-periodic waves of the* $(2+1)$*DBS equation*

In this section, we consider two-periodic wave solutions of Eq. (5.2). According to Theorem 2, $k_i$, $\rho_i$ and $\omega_i$ should satisfy the following system

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}^2}\big[-4\pi^2\langle 2n-\theta_1,k\rangle\langle 2n-\theta_1,\omega\rangle + 16u_0\pi^2\langle 2n-\theta_1,k\rangle\langle 2n-\theta_1,\rho\rangle + 16\pi^4\langle 2n-\theta_1,k\rangle^{3\langle 2n-\theta_1,\rho\rangle} + c\rangle\big]\\
&\quad \times e^{\pi i[\langle\tau(n-\theta_1),n-\theta_1\rangle + \langle\tau n,n\rangle]} = 0,
\end{aligned} \tag{5.21a}$$

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}^2}\big[-4\pi^2\langle 2n-\theta_2,k\rangle\langle 2n-\theta_2,\omega\rangle + 16u_0\pi^2\langle 2n-\theta_2,k\rangle\langle 2n-\theta_2,\rho\rangle + 16\pi^4\langle 2n-\theta_2,k\rangle^{3\langle 2n-\theta_2,\rho\rangle} + c\rangle\big]\\
&\quad \times e^{\pi i[\langle\tau(n-\theta_2),n-\theta_2\rangle + \langle\tau n,n\rangle]} = 0,
\end{aligned} \tag{5.21b}$$

$$\sum_{n\in\mathbb{Z}^2}\left[-4\pi^2\langle 2n-\theta_3,k\rangle\langle 2n-\theta_3,\omega\rangle+16u_0\pi^2\langle 2n-\theta_3,k\rangle\langle 2n-\theta_3,\rho\rangle+16\pi^4\langle 2n-\theta_3,k\rangle^{3\langle 2n-\theta_3,\rho\rangle}+c\rangle\right]$$
$$\times e^{\pi i[\langle\tau(n-\theta_3),n-\theta_3\rangle+\langle\tau n,n\rangle]}=0, \tag{5.21c}$$
$$\sum_{n\in\mathbb{Z}^2}\left[-4\pi^2\langle 2n-\theta_4,k\rangle\langle 2n-\theta_4,\omega\rangle+16u_0\pi^2\langle 2n-\theta_4,k\rangle\langle 2n-\theta_4,\rho\rangle+16\pi^4\langle 2n-\theta_4,k\rangle^{3\langle 2n-\theta_4,\rho\rangle}+c\rangle\right]$$
$$\times e^{\pi i[\langle\tau(n-\theta_4),n-\theta_4\rangle+\langle\tau n,n\rangle]}=0. \tag{5.21d}$$

By introducing the notations as

$$\begin{aligned}
&H=(h_{ij})_{4\times 4},\qquad b=(b_1,b_2,b_3,b_4)^T,\\
&h_{i1}=-4\pi^2\sum_{(n_1,n_2)\in\mathbb{Z}^2}\langle 2n-\theta_i,k\rangle\big(2n_1-\theta_i^1\big)\Im_i(n),\qquad h_{i2}=-4\pi^2\sum_{(n_1,n_2)\in\mathbb{Z}^2}\langle 2n-\theta_i,k\rangle\big(2n_2-\theta_i^2\big)\Im_i(n),\\
&h_{i3}=16\pi^2\sum_{(n_1,n_2)\in\mathbb{Z}^2}\langle 2n-\theta_i,k\rangle\langle 2n-\theta_i,\rho\rangle\Im_i(n),\qquad h_{i4}=\sum_{(n_1,n_2)\in\mathbb{Z}^2}\Im_i(n),\\
&b_i=-16\pi^4\sum_{(n_1,n_2)\in\mathbb{Z}^2}\langle 2n-\theta_i,k\rangle^3\langle 2n-\theta_i,\rho\rangle\Im_i(n),\\
&\Im_i(n)=\wp_1^{n_1^2+(n_1-\theta_i^1)^2}\wp_2^{n_2^2+(n_2-\theta_i^2)^2}\wp_3^{n_1n_2+(n_1-\theta_i^1)(n_2-\theta_i^2)},\\
&\wp_1=e^{\pi i\tau_{11}},\qquad \wp_2=e^{\pi i\tau_{22}},\qquad \wp_3=e^{2\pi i\tau_{12}},\quad i=1,2,3,4,
\end{aligned}\tag{5.22}$$

Eqs. (5.21) can be written as a linear system

$$H(\omega_1,\omega_2,u_0,c)^T=b. \tag{5.23}$$

Now solving this system, we get a two-periodic wave solution of Eq. (5.2)

$$u=u_0-2\partial_x^2\ln\vartheta(\xi_1,\xi_2,\tau), \tag{5.24}$$

where $\vartheta(\xi_1,\xi_2,\tau)$ and parameters $\omega_1$, $\omega_2$, $u_0$, $c$ are given by Eqs. (3.7) and (5.23), respectively. The other parameters $k_1$, $k_2$, $\rho_1$, $\rho_2$, $\varepsilon_1$, $\varepsilon_2$, $\tau_{11}$, $\tau_{12}$ and $\tau_{22}$ are free.

### 5.4. Feature and asymptotic property of two-periodic waves

The two-periodic wave solution (5.24) has a simple characterization that are similar to CDGSK equation. And some other characterizations are obtained as follows:

(i) It has $2N$ fundamental periods $\{\zeta_i,\ i=1,2,\dots,N\}$ and $\{\tau_i,\ i=1,2,\dots,N\}$ in $(\xi_1,\xi_2)$ with $\zeta_1=(1,0,\dots,0)^T,\dots,\zeta_N=(0,0,\dots,1)^T$. Its velocity of propagation is given by

$$\frac{dx}{dt}=\frac{k_1\omega_2-k_2\omega_1}{k_1\rho_2-k_2\rho_1},\qquad \frac{dy}{dt}=\frac{\omega_1\rho_2-\omega_2\rho_1}{k_1\rho_2-k_2\rho_1}. \tag{5.25}$$

(ii) Assuming that parameters satisfy a ratio relation

$$\frac{k_2}{k_1}=\frac{\rho_2}{\rho_1}=d\quad (d\text{ is a constant}), \tag{5.26}$$

we have

$$\omega_2\sim d\omega_1,\qquad \xi_2\sim d\xi_1,\qquad \vartheta(\xi_1,\xi_2)\sim\vartheta(\xi_1,d\xi_1). \tag{5.27}$$

It shows that the two-periodic wave is actually one-dimensional and it degenerates to one-periodic wave (see Fig. 6).

(iii) If parameters do not satisfy a ratio relation, i.e.,

$$\frac{k_2}{k_1}\neq\frac{\rho_2}{\rho_1}, \tag{5.28}$$

then for any time $t$, phase variables $\xi_1=d_1$ and $\xi_2=d_2$ ($d_1$, $d_2$ are constants) intersect at a unique point. This point moves in the $(x,y)$ plane with a constant speed as the time $t$ changes. Every two-periodic wave as in Fig. 7 is spatially periodic in two directions, but it need not be periodic in either $x$ or $y$ directions.

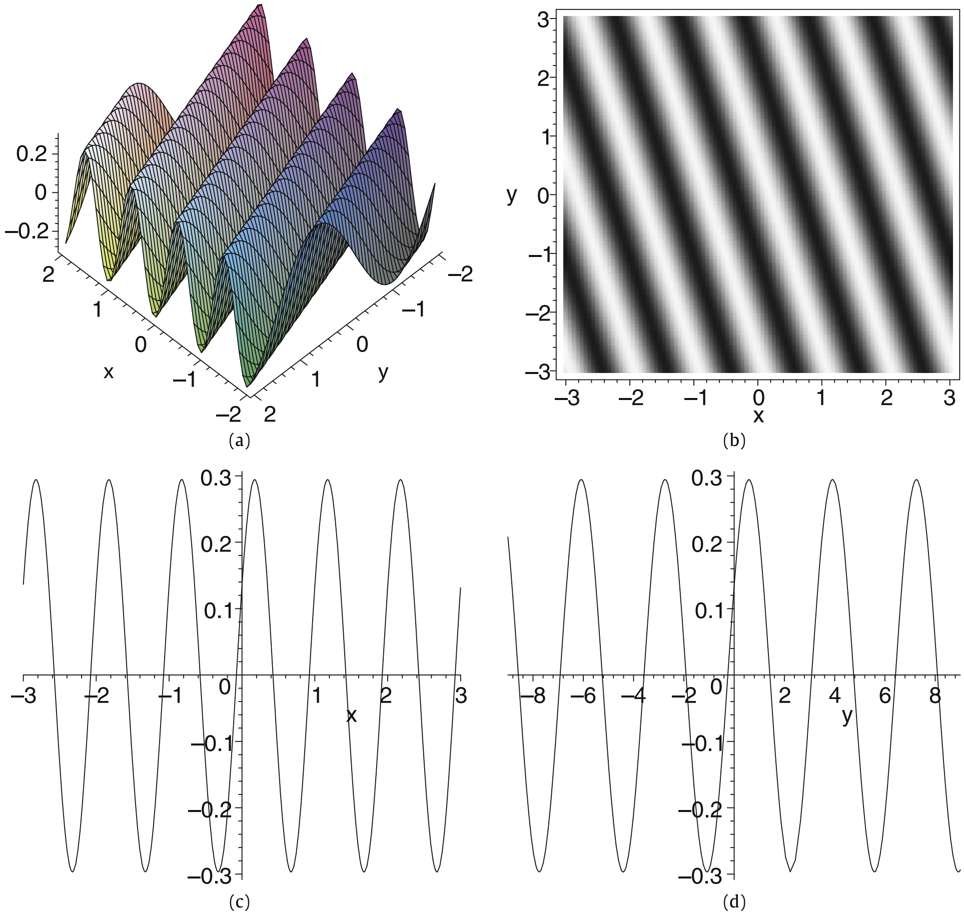



**Fig. 6.** (Color online) A degenerate two-periodic wave solution to the $(2+1)$DBS equation with parameters: $\frac{k_1}{k_2} = \frac{\rho_1}{\rho_2}$ and $k_1 = 0.1$, $k_2 = 0.3$, $\rho_1 = 0.2$, $\rho_2 = 0.6$, $\tau_{11} = 0.2i$, $\tau_{12} = 0.3i$, $\tau_{22} = i$, $\varepsilon_1 = \varepsilon_2 = 0$. This figure shows that the degenerate two-periodic wave is almost one-dimensional. (a) Perspective view of the wave. (b) Overhead view of the wave, with contour plot shown. The bright lines are crests and the dark lines are troughs. (c) Wave propagation pattern of the wave along the $x$ axis. (d) Wave propagation pattern of wave along the $y$ axis.

(iv) In a subcase of the (iv) $\tau_{11} = \tau_{12}$, $k_1 = k_2$, $\rho_1 = -\rho_2$, the two-periodic solution has only four independent parameters $k_1$, $\rho_1$, $\tau_{11}$ and $\tau_{12}$. The graph of $u$ is in Fig. 8. It is shown that the cell of its pattern is a regular hexagon from the contour plot (see Fig. 8(b)).

Finally, we consider the asymptotic properties of the two-periodic solution (5.24). In a way similar to Theorem 4, we can establish the relation between the two-periodic solution (5.24) and the two-soliton solution (5.5b) as follows.

**Theorem 6.** *If $(\omega_1, \omega_2, u_0, c)^T$ is a solution of the system* (5.23)*, and for the two-periodic wave solution* (5.24)*, we take*

$$k_i = \frac{\mu_i}{2\pi i}, \qquad \rho_i = \frac{\nu_i}{2\pi i}, \qquad \varepsilon_i = \frac{\delta_i + \pi\tau_{ij}}{2\pi i}, \qquad \tau_{12} = \frac{A_{12}}{2\pi i}, \quad i = 1, 2, \tag{5.29}$$

*where $\mu_i$, $\nu_i$, $\delta_i$, $i = 1, 2$, and $A_{12}$ are given in Eq.* (5.5b)*. Then we have the following asymptotic relations*

$$\begin{aligned} &u_0 \to 0, \qquad c \to 0, \qquad \xi_i \to \frac{\eta_i + \pi\tau_{ij}}{2\pi i}, \quad i = 1, 2, \\ &\vartheta(\xi_1, \xi_2, \tau) \to 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1 + \eta_1 + A_{12}}, \quad as\ \wp_1, \wp_2 \to 0. \end{aligned} \tag{5.30}$$

*It implies that the two-periodic solution* (5.24) *tends to the two-soliton solution* (5.5b) *under a small amplitude limit, that is* $(u, \wp_1, \wp_2) \to (u_1, 0, 0)$.

**Proof.** The proof of Theorem 6 is similar to the Theorem 4's and we do not give a detailed proof here. $\square$

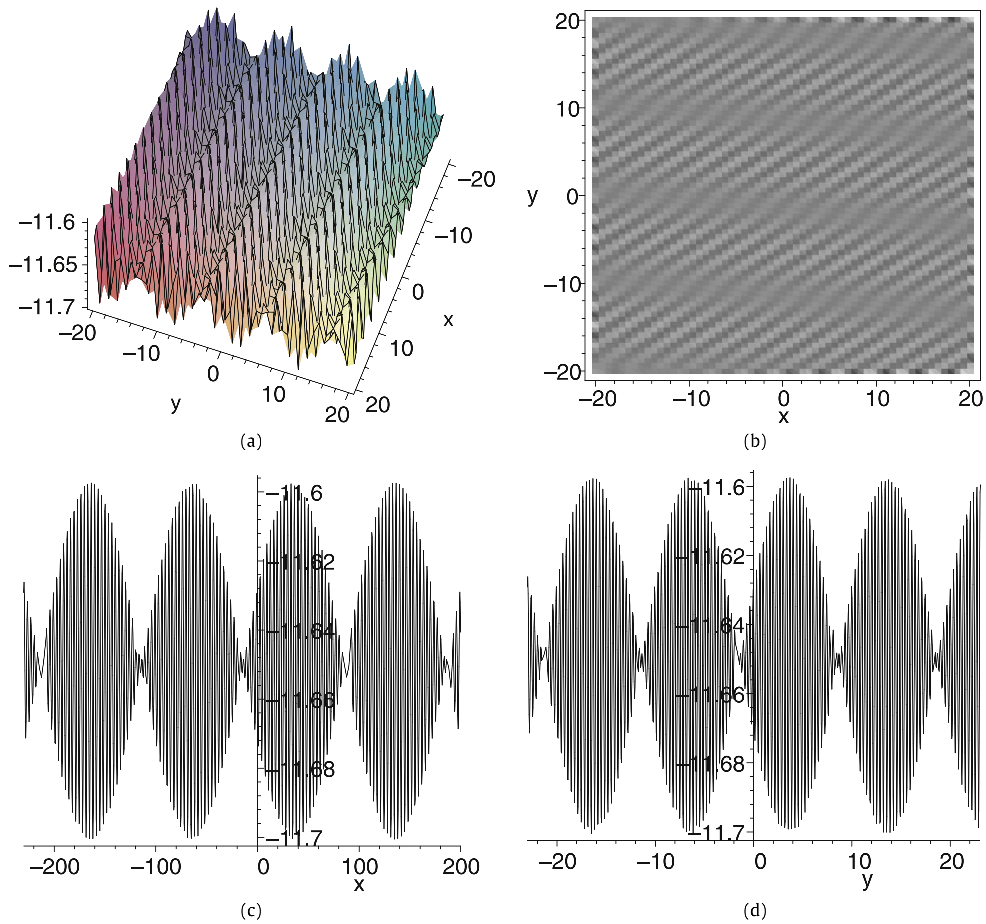



**Fig. 7.** (Color online) An asymmetric two-periodic wave to the $(2+1)$DBS equation with parameters: $\frac{k_1}{k_2} \neq \frac{\rho_1}{\rho_2}$ and $k_1 = 0.01$, $k_2 = 0.3$, $\rho_1 = 0.1$, $\rho_2 = -3$, $\tau_{11} = i$, $\tau_{12} = 0.5i$, $\tau_{22} = 2i$, $\varepsilon_1 = \varepsilon_2 = 0$. This figure shows that the asymmetric two-periodic wave is spatially periodic in two directions, but it need not be periodic in either the $x$ or $y$ directions. (a) Perspective view of the wave. (b) Overhead view of the wave, with contour plot shown. The bright small irregular quadrilaterals are crests and the dark small irregular quadrilaterals are troughs. (c) Wave propagation pattern of the wave along the $x$ axis. (d) Wave propagation pattern of wave along the $y$ axis.

## 6. Conclusion and discussions

In this paper, we obtain one Riemann and two Riemann theta functions periodic wave solutions of the $(1+1)$-dimensional CDGSK equation (4.1) and $(2+1)$-dimensional breaking soliton equation (5.2), which belong to the cases of $N = 1$ and $N = 2$, respectively. On this basis, we can also obtain these periodic wave solutions of other nonlinear equations. Once such an equation is written in a bilinear form, its periodic wave solutions can be directly obtained by using Theorems 1 and 2. By making a limiting procedure, we analyze asymptotic behavior of the multiperiodic waves in details and obtain the exact relations between the periodic wave solutions and the well-known soliton solutions, which is rigorously shown that the periodic wave solutions tend to the soliton solutions under a small amplitude limit.

Some other conclusions and discussions on our results are given as follows:

(i) Generalized Hirota–Riemann method is also applies to nonlinear differential equations such as Toeplitz lattice equation [44,45]

$$\frac{\partial u_n}{\partial s} = u_n(v_{n+1} - v_n), \qquad \frac{\partial v_n}{\partial t} = u_{n+1} - u_n, \tag{6.1a}$$

$$\frac{\partial u_n}{\partial t} = u_n(w_{n-1} - w_n), \qquad \frac{\partial w_n}{\partial s} = u_{n+1} - u_n. \tag{6.1b}$$

Obviously, system (6.1a) or system (6.1b) constitutes the two-dimensional Toda lattice. It is well known that (6.1a) or (6.1b) can be transformed into the same bilinear equation

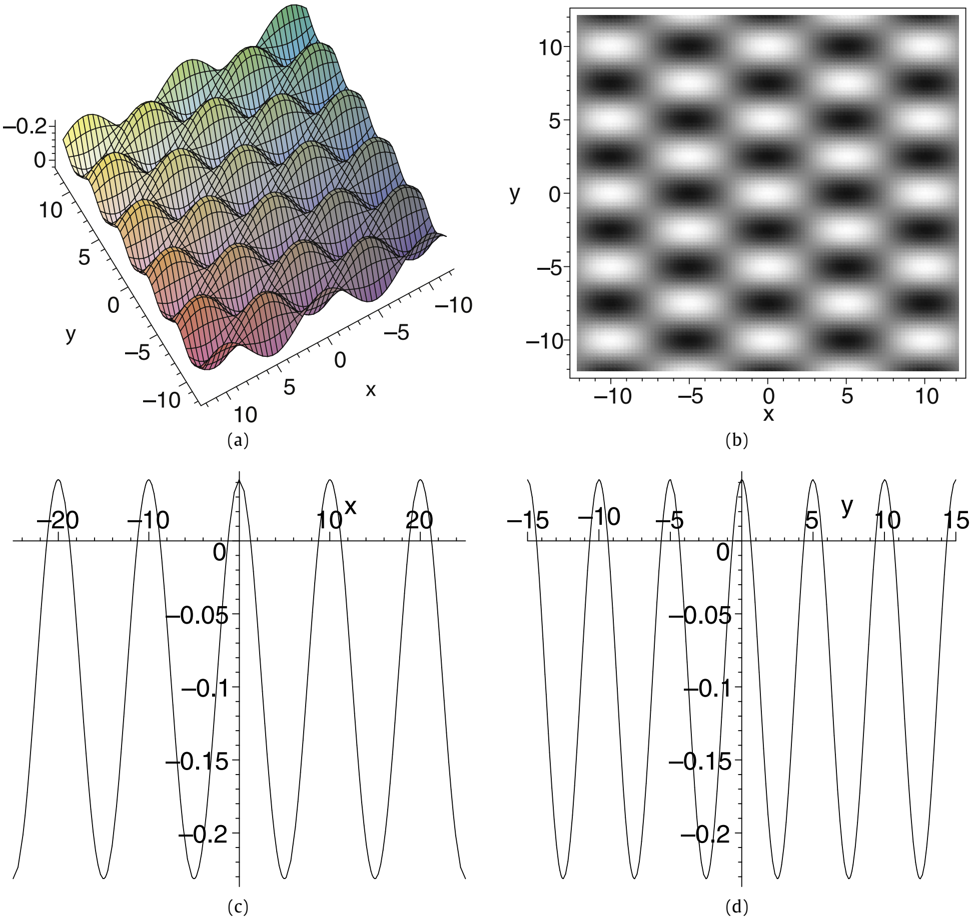



**Fig. 8.** (Color online) A symmetric two-periodic wave for the $(2+1)$DBS equation with parameters: $\frac{k_1}{k_2} \neq \frac{\rho_1}{\rho_2}$ and $k_1 = 0.1$, $k_2 = 0.1$, $\rho_1 = 0.2$, $\rho_2 = -0.2$, $\tau_{11} = i$, $\tau_{12} = 0.3i$, $\tau_{22} = i$, $\varepsilon_1 = \varepsilon_2 = 0$. This figure shows that the symmetric two-periodic wave is periodic in two directions. (a) Perspective view of the wave. (b) Overhead view of the wave, with contour plot shown. The bright hexagons are crests and the dark hexagons are troughs. (c) Wave propagation pattern of the wave along the $x$ axis. (d) Wave propagation pattern of wave along the $y$ axis.

$$\mathscr{L}(D_s, D_n, D_t) = \left(D_t D_x + 4\sinh^2\left(\frac{1}{2}D_n\right) + c\right) f_n \cdot f_n = 0, \tag{6.2}$$

by the dependent variable transformation

$$u_n = \frac{f_{n+1} f_n}{f_n^2}, \qquad v_n = \left(\ln \frac{f_n}{f_{n-1}}\right)_s, \qquad w_n = \left(\ln \frac{f_n}{f_{n+1}}\right)_t. \tag{6.3}$$

The following is similar to the nonlinear differential equation and we do not give a detailed proof here.

(ii) The results can be extended to the case when $N > 2$, but there are still certain numerical difficulties in the calculation. We only considered a condition for an $N$-periodic wave solution of nonlinear equation (2.1) in this section. The theta function takes the form

$$\vartheta(\xi, \tau) = \vartheta(\xi_1, \xi_2, \ldots, \xi_N, \tau) = \sum_{n \in \mathbb{Z}^2} e^{\pi i \langle \tau n, n\rangle + 2\pi i \langle \xi, n\rangle}, \tag{6.4}$$

where $n = (n_1, \ldots, n_N)^T \in \mathbb{Z}^N$, $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{C}^N$, $\xi_i = k_i x_1 + l_i x_2 + \cdots + \rho_i x_N + \omega_i t + \varepsilon_i$, $i = 1, 2, \ldots, N$, and $-i\tau$ is a positive definite and real-valued symmetric $N \times N$ matrix.

Similarly, the function (6.4) satisfies the bilinear equation (2.10), we obtain from (6.4) and (2.10)

$$\begin{aligned}
&\mathscr{L}(D_{x_1}, D_{x_2}, \ldots, D_N, D_t)\vartheta(\xi_1, \xi_2, \ldots, \xi_N, \tau) \cdot \vartheta(\xi_1, \xi_2, \ldots, \xi_N, \tau) \\
&\quad = \sum_{m,n \in \mathbb{Z}^N} \mathscr{L}\big(2\pi i\langle n-m, k\rangle, 2\pi i\langle n-m, l\rangle, \ldots, 2\pi i\langle n-m, \rho\rangle, 2\pi i\langle n-m, \omega\rangle\big) e^{2\pi i\langle \xi, m+n\rangle + \pi i(\langle \tau m, m\rangle + \langle \tau n, n\rangle)}
\end{aligned}$$

$$= \sum_{m'\in\mathbb{Z}^N} \left\{ \sum_{n\in\mathbb{Z}^N} \mathscr{L}\big(2\pi i\langle 2n-m',k\rangle, 2\pi i\langle 2n-m',l\rangle,\ldots,2\pi i\langle 2n-m',\rho\rangle, 2\pi i\langle 2n-m',\omega\rangle\big) \times e^{\pi i[\langle\tau(n-m'),n-m'\rangle+\langle\tau n,n\rangle]} \right\} e^{2\pi i\langle\xi,m'\rangle}$$

$$\triangleq \sum_{m'\in\mathbb{Z}^N} \tilde{\mathscr{L}}\big(m'_1,\ldots,m'_N\big)e^{2\pi i\langle\xi,m'\rangle} = \sum_{m'\in\mathbb{Z}^N}\tilde{\mathscr{L}}\big(m'\big)e^{2\pi i\langle\xi,m'\rangle}, \quad m'=m+n. \tag{6.5}$$

Shifting index $n$ as $n'=n-\delta_{ij},\ j=1,2,\ldots,N$, we can compute that

$$\begin{aligned}\tilde{\mathscr{L}}\big(m'\big) &= \tilde{\mathscr{L}}\big(m'_1,\ldots,m'_N\big) = \sum_{n\in\mathbb{Z}^N}\mathscr{L}\big(2\pi i\langle 2n-m',k\rangle, 2\pi i\langle 2n-m',l\rangle,\ldots,2\pi i\langle 2n-m',\rho\rangle, 2\pi i\langle 2n-m',\omega\rangle\big)\\ &\quad\times e^{\pi i[\langle\tau(n-m'),n-m'\rangle+\langle\tau n,n\rangle]}\\ &= \sum_{n\in\mathbb{Z}^N}\mathscr{L}\Bigg(2\pi i\sum_{i=1}^N\big[2n'_i-\big(m'_i-2\delta_{ij}\big)\big]k_i,\ldots,2\pi i\sum_{i=1}^N\big[2n'_i-\big(m'_i-2\delta_{ij}\big)\big]\rho_i,\\ &\quad 2\pi i\sum_{i=1}^N\big[2n'_i-\big(m'_i-2\delta_{ij}\big)\big]\omega_i\Bigg)e^{\pi i\sum_{i,k=1}^N[(n'_i+\delta_{ij})(n'_k+\delta_{kj})+(m'_i-n'_i-\delta_{ij})(m'_k-n'_k-\delta_{kj})]\tau_{ik}}\\ &= \tilde{\mathscr{L}}\big(m'_1,\ldots,m_i-2,\ldots,m'_N\big)e^{2\pi i(\sum_{i=1}^N\tau_{ij}m'_j-\tau_{jj})},\end{aligned} \tag{6.6}$$

where $\delta_{ij}$ representing Kronecker's delta. It now follows that if

$$\begin{aligned}\tilde{\mathscr{L}}(0,0,\ldots,0) &= \tilde{\mathscr{L}}(1,0,\ldots,0)=\tilde{\mathscr{L}}(0,1,\ldots,0)=\cdots=\tilde{\mathscr{L}}(0,0,\ldots,1,\ldots,0)=\cdots\\ &= \tilde{\mathscr{L}}(0,0,\ldots,1)=0,\end{aligned} \tag{6.7}$$

then $\tilde{\mathscr{L}}(m'_1,\ldots,m'_N)=0$ for all $m'_1,\ldots,m'_N\in\mathbb{Z}$ and thus the function (6.4) is an exact solution of Eq. (2.10).

(iii) Solution (2.10) reproduces the cnoidal waves as $N=1$, which is actually the Weierstrass or Jacobi elliptic solution (for example, see [46,47]) according to the following relations:

$$\wp(\xi,\tau) = -\ln\big(\vartheta_{11}(\xi,\tau)\big)''+c, \qquad \mathrm{cn}\big(\pi\vartheta_{11}(0,\tau),K\big)=\frac{\vartheta_{01}(0,\tau)\vartheta_{10}(\xi,\tau)}{\vartheta_{10}(0,\tau)\vartheta_{01}(\xi,\tau)}, \tag{6.8}$$

where $K=\frac{\vartheta_{10}(0,\tau)^2}{\vartheta_{00}(0,\tau)^2}$, $K$ is called the modulus of the Jacobi elliptic function and $c$ is defined so that the Laurent expansion of $\vartheta_{11}(\xi,\tau)$ at $\xi=0$ has a zero constant term. Four auxiliary theta functions are defined by

$$\vartheta_{00}(\xi,\tau)=\vartheta(\xi,\tau), \tag{6.9a}$$

$$\vartheta_{01}(\xi,\tau)=\vartheta\left(\xi+\frac{1}{2},\tau\right), \tag{6.9b}$$

$$\vartheta_{10}(\xi,\tau)=e^{\frac{\pi i\tau}{4}+\pi i\xi}\vartheta\left(\xi+\frac{1}{2}\tau,\tau\right), \tag{6.9c}$$

$$\vartheta_{11}(\xi,\tau)=e^{\frac{\pi i\tau}{4}+\pi i(\xi+\frac{1}{2})}\vartheta\left(\xi+\frac{1}{2}(1+\tau),\tau\right). \tag{6.9d}$$

The specific definition and their identities of these auxiliary theta functions are given in Appendices A and B, respectively.

In conclusion, used a multidimensional Riemann theta function, a lucid and straightforward way is presented to explicitly construct multiperiodic Riemann theta functions periodic wave solutions for these Hirota bilinear equations. We hope this method could help to better understand the diversity and integrability of nonlinear differential equations, which is also suitable for other more general nonlinear evolution equations in mathematical physics.

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## Appendix A. The theta functions

The theta functions $\theta_n(x)$, $n = 1, 2, 3, 4$, the parameters $q$ (the nome) and $\tau$ (pure imaginary) are defined by [48,49]

$$\vartheta_{0,0}(\xi) = \vartheta_{0,0}(\xi, \tau) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos(2nx) = \sum_{m=-\infty}^{\infty}\exp\bigl(\pi i\tau m^2 + 2imx\bigr), \tag{A.1}$$

$$\vartheta_{0,1}(\xi) = \vartheta_{0,1}(\xi, \tau) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos(2nx) = \sum_{m=-\infty}^{\infty}\exp\left(\pi i\tau m^2 + 2im\left(x + \frac{\pi}{2}\right)\right), \tag{A.2}$$

$$\vartheta_{1,0}(\xi) = \vartheta_{1,0}(\xi, \tau) = 2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2}\cos\bigl((2n+1)x\bigr) = \sum_{m=-\infty}^{\infty}\exp\left(\pi i\tau\left(m + \frac{1}{2}\right)^2 + 2i\left(m + \frac{1}{2}\right)x\right), \tag{A.3}$$

$$\begin{aligned}\vartheta_{1,1}(\xi) &= \vartheta_{1,1}(\xi, \tau) = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin\bigl((2n+1)x\bigr)\\ &= -\sum_{m=-\infty}^{\infty}\exp\left(\pi i\tau\left(m + \frac{1}{2}\right)^2 + 2i\left(m + \frac{1}{2}\right)\left(x + \frac{\pi}{2}\right)\right),\end{aligned} \tag{A.4}$$

where $0 < \mathrm{abs}(q) < 1$, $q = \exp(\pi i\tau) = \exp(-\frac{\pi K'}{K})$, $K = \frac{\theta_2^2(0)}{\theta_3^2(0)}$, $K' = \frac{\theta_4^2(0)}{\theta_3^2(0)}$ and $K^2 + (K')^2 = 1$. It is clear that $\vartheta_{1,1}(\xi, \tau)$ is an odd function while the other three are even functions. And the zeros of $\theta_1(x)$, $\theta_2(x)$, $\theta_3(x)$ and $\theta_4(x)$ are at $M\pi + N\pi\tau$, $(M + \frac{1}{2})\pi + N\pi\tau$, $(M + \frac{1}{2})\pi + (N + \frac{1}{2})\pi\tau$, $M\pi + (N + \frac{1}{2})\pi\tau$, $M, N \in \mathbb{Z}$, respectively.

## Appendix B. The theta functions identities

Theta functions possess a huge variety of product identities involving products of theta functions, e.g.,

$$\vartheta_{0,0}(x+y)\vartheta_{0,0}(x-y)\vartheta_{1,0}^2(0) = \vartheta_{0,1}^2(x)\vartheta_{1,1}^2(y) + \vartheta_{0,0}^2(x)\vartheta_{1,0}^2(y), \tag{B.1}$$

$$\vartheta_{0,1}(x+y)\vartheta_{0,1}(x-y)\vartheta_{1,0}^2(0) = \vartheta_{0,1}^2(x)\vartheta_{1,0}^2(y) + \vartheta_{0,0}^2(x)\vartheta_{1,1}^2(y), \tag{B.2}$$

$$\vartheta_{1,0}(x+y)\vartheta_{1,0}(x-y)\vartheta_{1,0}^2(0) = \vartheta_{1,0}^2(x)\vartheta_{1,0}^2(y) - \vartheta_{1,1}^2(x)\vartheta_{1,1}^2(y), \tag{B.3}$$

$$\vartheta_{1,1}(x+y)\vartheta_{1,1}(x-y)\vartheta_{1,0}^2(0) = \vartheta_{1,1}^2(x)\vartheta_{1,0}^2(y) - \vartheta_{1,0}^2(x)\vartheta_{1,1}^2(y), \tag{B.4}$$

$$\vartheta_{1,1}(x+y)\vartheta_{1,0}(x-y)\vartheta_{0,0}(0)\vartheta_{0,1}(0) = \vartheta_{1,1}(x)\vartheta_{1,0}(x)\vartheta_{0,0}(y)\vartheta_{0,1}(y) + \vartheta_{0,0}(x)\vartheta_{0,1}(x)\vartheta_{1,1}(y)\vartheta_{1,0}(y), \tag{B.5}$$

$$\vartheta_{0,0}(x+y)\vartheta_{0,1}(x-y)\vartheta_{0,0}(0)\vartheta_{0,1}(0) = \vartheta_{0,0}(x)\vartheta_{0,1}(x)\vartheta_{0,0}(y)\vartheta_{0,1}(y) - \vartheta_{1,1}(x)\vartheta_{1,0}(x)\vartheta_{1,1}(y)\vartheta_{1,0}(y). \tag{B.6}$$

## Appendix C. The definition of Hirota bilinear operator

The Hirota bilinear operators $D_{x_1}, D_{x_2}, \ldots, D_t$ and $D_n$ are defined as follows:

$$e^{\delta D_n} f(n)\cdot g(n) = e^{\delta(\partial_n - \partial_{n'})}\big|_{n'=n} = f(n+\delta)g(n-\delta), \tag{C.1}$$

$$\cosh(\delta D_n) f(n)\cdot g(n) = \frac{1}{2}\bigl(e^{\delta D_n} + e^{-\delta D_n}\bigr) f(n)\cdot g(n), \tag{C.2}$$

$$\sinh(\delta D_n) f(n)\cdot g(n) = \frac{1}{2}\bigl(e^{\delta D_n} - e^{-\delta D_n}\bigr) f(n)\cdot g(n), \tag{C.3}$$

$$D_{x_1}^m D_{x_2}^n \cdots D_{x_N}^p D_t^r f(X,t)\cdot g(X,t) = (\partial_{x_1} - \partial_{x_1'})^m(\partial_{x_2} - \partial_{x_2'})^n\cdots(\partial_{x_N} - \partial_{x_N'})^p(\partial_t - \partial_{t'})^r f(X,t)\cdot g\bigl(X', t'\bigr)\big|_{X=X',\,t=t'}, \tag{C.4}$$

where $X = (x_1, x_2, \ldots, X_N)$ and $X' = (x_1', x_2', \ldots, X_N')$.

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